Georgia Tech High School Math Competition

Multiple Choice Test

February 15, 2014

- Each correct answer is worth one point; there is no deduction for incorrect answers.
- Make sure to enter your ID number on the answer sheet.
- You may use the test booklet as scratch paper, but no credit will be given for work in the booklet.
- You may keep the test booklet after the test has ended.

Problems contributed by Albert Bush, Tongzhou Chen, Santhosh Karnik, Robert Krone, Chris Pryby, Shane Scott, Peter Whalen, Peter Woolfitt, and Chi-Ho Yuen. Thanks also to Tobias Hurth and Prof. Tom Morley for assistance in editing.

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1. Derek forms three 3-digits numbers a, b and c using the digits 1 to 9 exactly once each, what is the largest possible value he can get for $a + b - c$?

   (a) 333

   (b) 1320

   (c) 1518

   (d) 1716

   (e) 1839

Solution: D. We want to maximize a and b while minimizing c. To maximize a and b we use the largest digits, $4, 5, 6, 7, 8, 9$ and use $8, 9$ for the hundreds-places, $6, 7$ for the tens-places and $4, 5$ for the ones. For example $a = 975$, $b = 864$. Similarly let $c = 123$. The outcome is 1716.

2. 10 people each own an umbrella. Every person is handed one of these umbrellas chosen at random. What is the probability of exactly 9 people getting their own umbrella?

   (a) 0

   (b) 1/10

   (c) $9/10^{10}$

   (d) 1/10!

   (e) 1

Solution: A. If 9 people each get their own umbrella, the tenth person must also get their own umbrella, so it is impossible for exactly 9 people to get their own umbrella.

3. Ed and Ted are playing a series of tennis games, and they switch sides of the court after every odd-numbered game. Ted forgets the score in the middle of a game, but he knows he is on the same side as he was in the first game. If the score is written x—y, where x is the number of games Ed has won and y is the number Ted has won, which of the following could *not* be the current score?

   (a) 4—1

   (b) 2—2

   (c) 4—3

   (d) 5—2

(e) All of these could be possible game scores.

Solution: A. Since the players switch sides after odd-numbered games, Ed and Ted will be on the same side as their first game during game 1; on opposite sides during games 2 and 3; on the original sides during games 4 and 5; and so forth. In other words, during the nth game, then the players are on the original sides whenever $n \equiv 0, 1 \pmod 4$. If the current game score is x—y, then $x + y = n - 1$, so the players are on the original sides when $x + y \equiv 0, 3 \pmod 4$. The only choice above for which $x + y \not\equiv 0, 3 \pmod 4$ is 4—1.

4. 7^3 unit cubes are stacked into a $7 \times 7 \times 7$ cube. How many of the small cubes are on the surface of the large cube?
   (a) 127
   (b) 134
   (c) 218
   (d) 294
   (e) 327

Solution: C. The cubes not on the surface form a $5 \times 5 \times 5$ cube, so there are 125 of them. There are $343 - 125 = 218$ cubes on the surface.

5. A piece of paper is in the shape of a semicircle with radius 1. It is formed into a cone by taping together the two exposed radii. Find the height of the cone.
   (a) $\sqrt{3}/2$
   (b) $\sqrt{2}/2$
   (c) π
   (d) $\sqrt{\pi^2 - 1}$
   (e) $\sqrt{3}$

Solution: A. The circumference of the base of the cone is π, so the radius of the base is $1/2$. Drawing the altitude of the cone produces a right triangle with hypotenuse 1 and one side $1/2$. The height is the remaining side which has length $\sqrt{3}/2$

6. Peter and Santhosh play a game where they take turns flipping a fair coin, starting with Peter. If a player gets heads, he wins. Otherwise he passes the coin to the other player and the game continues. What is the probability that Peter wins the game?
   (a) 1/3
   (b) 1/2
   (c) 2/3
   (d) 3/4
   (e) 4/5

Solution: C. The probability that Peter wins on the first flip is 1/2. The probability that Peter flips tails and then Santhosh wins on the second flip is $1/2 \cdot 1/2 = 1/4$. The probability that Peter flips tails, then Santhosh flips tails, then Peter wins on the third flip is $1/2 \cdot 1/2 \cdot 1/2 = 1/8$. In this way, we observe that Peter has twice the probability of winning as does Santhosh. Since their probabilities of winning must add up to 1, if p is the probability of Peter's winning, we must have $p + p/2 = 1$, so $p = 2/3$.

7. The *twisted cubic* is a curve in $\mathbb{R}^3$ defined by the equations $y = x^2$ and $z = x^3$. Find the area of the triangle whose vertices are the intersection of the twisted cubic with the plane $3x + 2y - z = 0$.
   (a) $6\sqrt{14}$
   (b) $12\sqrt{14}$
   (c) 24
   (d) 48
   (e) None of the above

Solution: A. To find the intersection points, substitute x^2 for y and x^3 for z in the equation for the plane, giving $3x + 2x^2 - x^3 = 0$, or $x(x+1)(x-3) = 0$. This means the intersections occur at the origin, $(-1, 1, -1)$, and $(3, 9, 27)$. The area of the triangle formed by these points is half the magnitude of the cross product of the vectors $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 9 \\ 27 \end{bmatrix}$. Since the cross product is

$$\begin{bmatrix} (1 \cdot 27 - (-1) \cdot 9) \\ -((-1) \cdot 27 - (-1) \cdot 3) \\ ((-1) \cdot 9 - 1 \cdot 3) \end{bmatrix} = \begin{bmatrix} 36 \\ 24 \\ -12 \end{bmatrix} = 12 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix},$$

the area of the triangle is

$$\frac{12}{2}\sqrt{9+4+1} = 6\sqrt{14}.$$

8. A rectangle is inscribed within a square of side length 1, such that each vertex of the rectangle lies on a different side of the square. The rectangle has length twice its width. Find the area of the rectangle.

   (a) 2/9

   (b) 4/9

   (c) 2/3

   (d) $\sqrt{2}/3$

   (e) $2\sqrt{2}/3$

Solution: B. The only way to inscribe a rectangle into a square such that each vertex lies on a different side is as in the figure below, such that four right isosceles triangles are made from the corners of the square. The smaller isosceles triangles have hypotenuse x and leg lengths $x/\sqrt{2}$, and the larger triangles have hypotenuse $2x$ and leg lengths $\sqrt{2}x$. Since $x/\sqrt{2} + \sqrt{2}x = 1$, we conclude that $x = \sqrt{2}/3$. The area of the rectangle is therefore $2x^2 = 4/9$.

9. Chris plays a game where he has 12 chips which he can divide up as he chooses into pile A and pile B. After Chris places the chips, Robert gives him a score which is the smaller of the following two values:

   - the square of the number of chips in pile A or
   - twice the number of chips in pile B.

   How many chips should Chris put in pile A in order to maximize his score?

   (a) 4

   (b) 5

   (c) 6

   (d) 8

   (e) 12

Solution: A. Let x be the number of chips placed on A. Chris' score is $\min\{x^2, 2(12-x)\}$. The maximum occurs when the two values are equal, so $x^2 = 2(12-x)$. The positive solution is $x = 4$.

10. Bag A contains marbles uniquely labeled 1 through 3. Bag B contains marbles uniquely labeled 4 through 10. Bag C contains marbles uniquely labeled 11 through 21. One marble is drawn from each bag. What is the most likely value of the sum?

    (a) $24, 25$, and 26 are equally the most likely

    (b) 24 is the most likely

    (c) 25 is the most likely

    (d) $23, 24, 25, 26$ and 27 are equally the the most likely

    (e) 23 and 24 are equally the most likely

Solution: A. Consider the marble values respectively as a, b, and c. The possible values of $a+b$ range from 5 to 13. Given any such value, c can be chosen uniquely to obtain $24, 25$, or 26, so these three outcomes each have 15 ways of occurring. For any other sum value there are some values of $a+b$ which are forbidden, so the number of ways of producing it is strictly smaller.

11. There are 8 players in a doubles ping-pong competition. In how many different ways can they be divided up into 4 teams of 2 players each? (The teams are not labeled.)

    (a) 16

    (b) 105

    (c) 1260

    (d) 2520

    (e) 40320

Solution: B. If the teams were named, there would be $8!/(2!)^4$ ways to assign each team two of the players. However, this over counts by a factor of 4! since we can reorder the four teams without changing how the players are grouped, so the result is

$$\frac{8!}{4!(2!)^4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{2^4} = 105.$$

12. Find the smallest value of $a^2 + b^2 + c^2 + d^2$ such that $a + 2b + 3c + 4d = 60$.
    (a) 36
    (b) 60
    (c) 72
    (d) 120
    (e) 144

Solution: D. Let $\vec{v} = (a, b, c, d)$ and $\vec{w} = (1, 2, 3, 4)$. We would like to find the vector $\vec{v}$ satisfying $\vec{v} \cdot \vec{w} = 60$ while minimizing $|\vec{v}|^2$. This happens when $\vec{v}$ and $\vec{w}$ point in the same direction, so $\vec{v} = k\vec{w}$ for some scalar k. Solving $k\vec{w} \cdot \vec{w} = 60$, we find $k = 2$, so $\vec{v} = (2, 4, 6, 8)$ and $|\vec{v}|^2 = 120$.

13. Count how many triples (x, y, z) of positive integers satisfy $x + y + z = 2014$.
    (a) 2021055
    (b) 2025078
    (c) 2027091
    (d) 2031120
    (e) 1359502364

Solution: B. This is equivalent to asking how many ways we can partition 2014 objects in three categories, such that no category is empty. Using the stars and bars technique, there are 2011 stars (since each category must have at least 1) and two bars. The number of ways to arrange the stars and bars is

$$\binom{2011+2}{2} = 2025078.$$

14. There are 7 pigeon holes arranged in a line and two pigeons that sleep in these holes at night. On the first day, each pigeon picks a hole independently and uniformly at random to stay in. Every day after the first each pigeon moves to a hole adjacent to the one it slept in the night before. If the pigeon has a choice whether to move left or right, it moves in either direction with probability 1/2. What is the probability that the two pigeons never spend a night in the same hole?
    (a) 1/2

(b) 33/49

(c) 24/49

(d) 1/4

(e) 0

Solution: C. Label the holes in order 1 through 7. Then this is really a question of parity - note that each day the parity of the hole that a pigeon stays in alternates. The chances of the two pigeons starting in holes of the same parity is $(3/7)^2 + (4/7)^2 = 25/49$. If the pigeons start in holes of the same parity, then eventually they are guaranteed to meet. Hence the answer is 24/49.

15. You live in a house with footprint in the shape of a regular hexagon of side length 1. The sprinkler system covers all the ground outside within distance 1 of the house. What is the area covered by the sprinklers?

(a) $6 + \pi$

(b) $9\sqrt{3}/2$

(c) $6 + 2\sqrt{3}$

(d) $3\sqrt{3} + 2\pi$

(e) 7

Solution: A. The area can be broken down into 6 squares along each side of the house, and 6 disk slices at each corner. The disk slices add up to one full disk with area π, while the squares have total area 6, for a total of $6 + \pi$. (In fact for any convex shape house the area will be perimeter $+ \pi$.)

16. Sharon means to be truthful, however whenever she tries to say an integer, she says a different one than what she means. Sharon makes the following statements:

- "Whenever I say an integer, I say the integer 2 more than what I mean."
- "My favorite integer is 2 more than 5."

What is Sharon's favorite number?

(a) 3

(b) 4

(c) 5

(d) 6

(e) 7

Solution: C. When Sharon says the number "2" in the first statement, she actually means the integer x with $x + x = 2$, so $x = 1$. Therefore she means to say, "My favorite number is 1 more than 4", which gives the answer 5.

17. Let f be an infinitely-differentiable function for all $x > 0$. Suppose $f(1) = -1$ and that $f'(x) = f(x)^2$. If $f^{(2014)}(x)$ is the 2014$^{\text{th}}$ derivative of f at x, what is $f^{(2014)}(1)$?

(a) $-2014!$

(b) -2014

(c) 1

(d) 2014

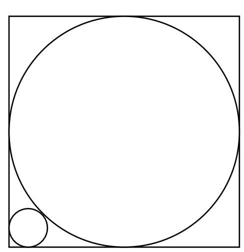
(e) 2014!

Solution: A. The n^{th} derivative of f is equal to $n!f^{n+1}$. This holds by induction: $f' = f^2$, and if it is true that $f^k = k!f^{k+1}$ for all $1 \leq k < n$, then using the power and chain rules we obtain

$$f^{(n)} = [f^{(n-1)}]' = [(n-1)!f^n]' = n!f^{n-1}f' = n!f^{n+1}.$$

Thus, $f^{(2014)}(1) = 2014!f(1)^{2015} = -2014!$.

18. A circle is inscribed in a square with side length 2. A smaller circle is drawn which is tangent to the first circle, and tangent to exactly two sides of the square. Find the radius of the smaller circle.



(a) $(\sqrt{2} - 1)/3$

(b) $(\sqrt{2}-1)/2$

(c) $\sqrt{2}/8$

(d) $3-2\sqrt{2}$

(e) $1/6$

Solution: D. The distance between the point of tangency between the two circles and the corner of the square is $\sqrt{2}-1$. This distance is $\sqrt{2}+1$ times the radius of the small circle, r, so

$$r = \frac{\sqrt{2}-1}{\sqrt{2}+1} = \frac{(\sqrt{2}-1)^2}{2-1} = 3-2\sqrt{2}.$$

19. A meter stick is broken in two places chosen independently and uniformly at random along the length. What is the probability that the three pieces can be formed into a triangle?

(a) $1/8$

(b) $1/6$

(c) $1/4$

(d) $1/3$

(e) $1/2$

Solution: C. To satisfy the triangle inequality, no piece may be longer than $1/2$. Let a and b be the two break points. If $a, b > 1/2$ the first piece will be too long, which has probability $1/4$. Similarly for $a, b < 1/2$. Finally, the middle piece is too long if $|b-a| > 1/2$, which also has probability $1/4$, leaving the answer as (C).

20. Three friends, Alice, Bob, and Carol, are driving across the country. They travel the same distance, but each of their cars requires refueling after a different number of miles. Alice's car is refueled every 160 miles, Bob's car is refueled every 240 miles, and Carol's car is refueled every 400 miles.

They all start with full gas tanks, but Alice finishes with an empty tank, while Bob finishes with 80 miles worth of gas remaining, and Carol finishes with 160 miles worth of gas remaining. Given that the trip is between 2000 and 4000 miles long, what is the total length of the trip?

(a) 2400 miles

(b) 2480 miles

(c) 2880 miles

(d) 3040 miles

(e) 3200 miles

Solution: D. We'll work in units of 80 miles to make the arithmetic simpler; let x be the total length of the trip in these units. So Alice can drive 2 units before refueling, Bob can drive 3 units, and Carol can drive 4 units. At the end of the trip, Alice must have driven 2 unit since her last refueling, Bob must have driven 2 units since his last refueling, and Carol must have driven 3 units since her last refueling. So $x \equiv 0 \pmod 2$, $x \equiv 1 \pmod 3$, and $x \equiv 2 \pmod 5$.

By the Chinese remainder theorem, this means that $x \equiv 8 \pmod{30}$. Interpreting this in terms of miles, the total distance must be $640 + 2400k$ miles, where k is some nonnegative integer. Given that the trip must be between 2000 and 4000 miles long, the only possibility is $k = 1$, giving 3040 miles.

21. Suppose a laser is on the edge of a circular room which has its wall covered in perfectly reflective mirror. At how many directions can the laser be shot so that the beam hits exactly 2014 distinct points on the room's wall, including the point where the laser is mounted? (Ignore vertical motion. Assume the beam travels parallel to the floor.)

(a) 420

(b) 936

(c) 1007

(d) 2014

(e) Infinitely many

Solution: B. Let θ be the angle between the origin of the laser and the first place it hits the wall. The points hit by the laser are at angles $\theta, 2\theta, 3\theta, \ldots$. The first time the laser returns to a point it has previously hit, it will repeat the previous path and never hit any new points. Therefore the 2015th point must coincide with a previous point, and in fact it must be the first point or else the 2014th point also wouldn't be new. Therefore $\theta = 2\pi n/2014$ for some $0 < n < 2014$. n and 2014 must be relatively prime or else the number of points will be strictly less than 2014.

The number of such values for n is the Euler totient function $\phi(2014) = (2-1)(19-1)(53-1) = 936$. We can also compute this value using inclusion-exclusion to count how many integers $0 < n < 2014$ are divisible by none of 2, 19 and 53.

22. How many integers $1 \leq n \leq 50$ are there so that the least common multiple of $1, 2, ..., n$ is the same as the least common multiple of $1, 2, ..., n+1$?

    (a) 43

    (b) 35

    (c) 31

    (d) 30

    (e) 27

Solution: E. n does not satisfy the condition iff $n+1$ is a power of prime, there are 15 primes from 2 to 51, and 8 more strict powers.

23. The number $x = \sqrt[3]{7+\sqrt{22}} + \sqrt[3]{7-\sqrt{22}}$ is a root of which monic cubic polynomial with integer coefficients?

    (a) $x^3 - 9x - 14$

    (b) $x^3 + 9x - 14$

    (c) $x^3 + 9x^2 - 14$

    (d) $x^3 + 9x^2 + 14$

    (e) None of the above

Solution: A. Let $a = \sqrt[3]{7+\sqrt{22}}$ and $b = \sqrt[3]{7-\sqrt{22}}$. Then

$$x^3 = (a+b)^3 = (a^3+b^3) + 3(a^2b + ab^2) = (a^3+b^3) + 3abx.$$

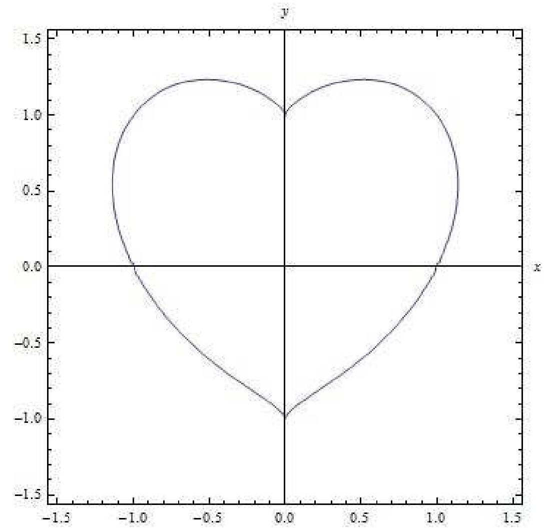
Note that

$$a^3 + b^3 = 7 + \sqrt{22} + 7 - \sqrt{22} = 14,$$

$$ab = \sqrt[3]{(7+\sqrt{22})(7-\sqrt{22})} = \sqrt[3]{7^2 - 22} = \sqrt[3]{27} = 3.$$

Thus, x satisfies $x^3 = 14 + 9x$, or $x^3 - 9x - 14 = 0$.

24. Let A_1 be the area of the region in the xy-plane bounded by the curve $(x^2+y^2-1)^3 - x^2y^3 = 0$ (graphed below) and let A_2 be the area of the region in the xy-plane bounded by the curve $(x^2+12y^2-4)^3 - 24x^3y^2 = 0$. Compute $\dfrac{A_1}{A_2}$.



(a) $\dfrac{2}{3}$

(b) $\dfrac{\sqrt{3}}{2}$

(c) 1

(d) $\dfrac{2\sqrt{3}}{3}$

(e) $\dfrac{3}{2}$

Solution: B. Note that the equation for the second curve can be manipulated as follows:

$$\begin{aligned}
0 &= (x^2+12y^2-4)^3 - 24x^3y^2 \\
&= 4^3\left(\frac{x^2}{4}+3y^2-1\right)^3 - 24x^3y^2 \\
&= 64\left(\left(\frac{x}{2}\right)^2 + (y\sqrt{3})^2 - 1\right)^3 - 64\left(\frac{x}{2}\right)^3 (y\sqrt{3})^2 \\
&= 64\left[\left((y\sqrt{3})^2 + \left(\frac{x}{2}\right)^2 - 1\right)^3 - (y\sqrt{3})^2\left(\frac{x}{2}\right)^3\right].
\end{aligned}$$

Hence, the second region can be obtained by the change of coordinates which sends (x, y) to $(y\sqrt{3}, x/2)$. Hence, $A_2 = 2 \cdot \frac{1}{\sqrt{3}} \cdot A_1$, and so, $\frac{A_1}{A_2} = \frac{\sqrt{3}}{2}$.

25. The set of *Gaussian integers* consists of numbers of the form $a + bi$, where a and b are integers and $i = \sqrt{-1}$. A Gaussian integer is called a *Gaussian prime* if it cannot be written as the product of two other Gaussian integers (not counting products in which one of the two numbers is 1, i, -1, or $-i$).

    Which of the following is a Gaussian prime?

    (a) 2

    (b) 5

    (c) 7

    (d) 13

    (e) All of the above

Solution: C. All four choices are prime numbers. A real prime p has no non-trivial real factors. If p is not a Gaussian prime, then it must be a product two complex Gaussian integers and moreover the factors must be of the form $(a + bi)$ and $(a - bi)$ where a and b are relatively prime, or else p would have a real factor. So then $p = a^2 + b^2$. It can be exhaustively checked that $2 = 1^2 + 1^2$, $5 = 1^2 + 2^2$, $13 = 2^2 + 3^2$, but 7 cannot be written as the sum of two integer squares.

26. How many triples of real numbers (a, b, c) are there so that each of the following three equations in x has exactly one real solution?

$$\frac{x^2}{4} + ax - b - c + 1 = 0,$$

$$\frac{x^2}{4} + bx - c - a + 1 = 0,$$

$$\frac{x^2}{4} + cx - a - b + 1 = 0.$$

    (a) 2

    (b) 3

    (c) 4

(d) 5

(e) 6

Solution: D. Each quadratic has exactly one solution when its discriminant is 0. This gives us the system of equations $a^2+b+c-1=0, a+b^2+c-1=0, a+b+c^2-1=0$. Subtracting the first equation from the second (and similarly for other pairs) we have $a(a-1)=b(b-1)=c(c-1)$. Therefore either $a=b$ or $a=1-b$, and similar relations hold for c. Either $a=b=c$, or two are equal and one is different, for instance $a=b=1-c$. In the first case, $a^2+2a-1=0$ which gives two solutions $(-1+\sqrt{2},-1+\sqrt{2},-1+\sqrt{2})$ and $(-1-\sqrt{2},-1-\sqrt{2},-1-\sqrt{2})$. In the second case, $a^2+a+(1-a)-1=0$ which gives the solution $(0,0,1)$ along with its permutations $(0,1,0)$ and $(1,0,0)$.

27. Let ΔABC be a triangle with $\angle B=85°, \angle C=45°$. The angle bisector at B meets AC at P and the angle bisector at C meets AB at Q. Find the angle $\angle APQ$.

(a) 32.5°

(b) 35°

(c) 47.5°

(d) 50.5°

(e) 55°

Solution: C. Let I to be the incenter of the triangle. We have $\angle AQI=102.5°$ and $\angle API=77.5°$, so A,P,I,Q are concyclic. Now $\angle APQ=\angle AIQ=180°-\angle IAQ-\angle AQI=47.5°$.

28. You have 12 balls numbered 1 to 12 and you have 12 boxes numbered 1 to 12. In how many ways can you put the balls into the boxes, one ball in each box, so that for each box, the number on the box and the number on the ball inside have no common factors other than 1?

(a) 7056

(b) 15876

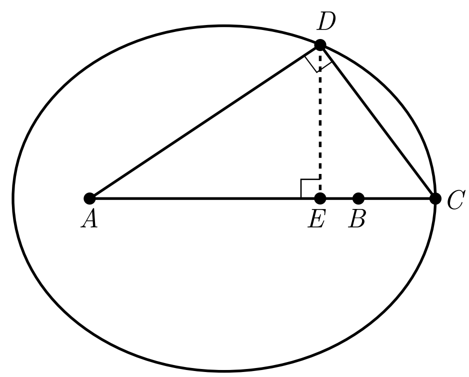
(c) 46656

(d) 63504

(e) 82944

Solution: D. The first observation is an odd box must contain an even ball and vice versa. So it suffices to count the number of ways to put ball 2,4,6,8,10,12 in box 1,3,5,7,9,11, and square this number. Ball 2,4,8 can be placed anywhere so the number of ways to place them after fixing the position of ball 6,10,12 is $3! = 6$; there are 12 ways to place ball 6,12 to avoid box 3,9, six of them avoid box 5 so ball 10 has 3 possible boxes to be placed, otherwise ball 10 has 4 possible boxes to be placed. Therefore the answer is $(3!(6 \times 3 + 6 \times 4))^2 = 63504$.

29. An ellipse has foci A and B with $AB = 4$. The segment $\overline{AB}$ is extended to point C on the ellipse and $BC = 1$. Point D also lies on the ellipse and $\angle ADC$ is a right angle. Let point E be the base of the altitude of triangle ADC from point D. Compute the length AE. (The diagram below is not necessarily to scale.)



(a) $5/4$

(b) $\sqrt{5}$

(c) 2

(d) $5\sqrt{3}/4$

(e) $3\sqrt{5}/2$

Solution: A. Let $x = AE$ and $EC = 5 - x$. Using similar triangles, the altitude DE is the geometric mean $\sqrt{x(5-x)}$. The major axis of the ellipse is 3, so $AD + BD = 2 \cdot 3$. Using the Pythagorean theorem, $AD = \sqrt{AE^2 + DE^2} = \sqrt{x^2 + x(5-x)}$ and $BD = \sqrt{BE^2 + DE^2} = \sqrt{(4-x)^2 + x(5-x)}$. Solving the following equation for x,

$$\sqrt{x^2 + x(5-x)} + \sqrt{(4-x)^2 + x(5-x)} = 6$$

the solutions are $x = 5$ and $x = 5/4$. We can throw out $x = 5$ since this is the degenerate case where $C = D = E$.

30. A sequence of integers $..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...$ satisfies

$$a_n + a_{-n} = a_{2n},$$

$$a_n - a_{-n} = a_{2n-1}$$

for all integers n, and $a_1 = 1$. Compute a_{2014}.

(a) -2

(b) -1

(c) 0

(d) 1

(e) 2

Solution: B. From the first rule we see that $a_{2n} = a_{-2n}$ for all n. From this and the second rule we can deduce that $a_{4n-1} = 0$ for all n. Finally we have $a_{4n+1} = a_{2n+1} - a_{-(2n+1)}$ for all n. Exactly one of $2n+1$ and $-(2n+1)$ is $3 \pmod 4$ so that term is 0, while the other is $1 \pmod 4$. As a result a_{4n+1} is either equal to a_{2n+1} or to $-a_{-(2n+1)}$ depending on which index is $1 \pmod 4$. Note that $|2n+1| < |4n+1|$ for all $n \neq 0$. By induction on the absolute value of the index that $a_{4n+1} = 1$ when $n \geq 0$ and $a_{4n+1} = -1$ when $n < 0$.

$$a_{2014} = a_{1007} + a_{-1007} = 0 + (-1) = -1$$

since $1007 \equiv 3 \pmod 4$.

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