1. A pair (x, y) of positive integers is called *happy* if $x+y$ and xy are both perfect squares. For example, $(5, 20)$ is happy because $5 + 20 = 25$ and $5 \cdot 20 = 100$.

   Prove that no happy pair exists in which one of its members is 3.

   Solution: Suppose there exists an x such that $(3, x)$ is happy. Then $3x$ is square, so $x = 3y^2$ for some y; then $3 + 3y^2$ is also square. However, the squares modulo 4 are 0 and 1, so $y^2 \equiv 0$ or 1 mod 4. This means that $3 + 3y^2 \equiv 2$ or 3 modulo 4, contradicting the fact that it is square.

2. Let $a_1, a_2, ..., a_{2013}$ be a sequence of 2013 positive integers. Prove that there exists a nonempty set of consecutive elements of the sequence whose sum is divisible by 2013.

   Solution: Define the partial sums $s_k = \sum_{i=1}^{k} a_i$ for $0 \le k \le 2013$. Any sum of consecutive elements can be expressed in terms of the partial sums as

   $$a_{j+1} + a_{j+2} + \cdots + a_i = s_i - s_j$$

   for some $i > j$. Note that $s_i - s_j$ is divisible by 2013 exactly when $s_i \equiv s_j \pmod{2013}$. There are 2014 partial sums but only 2013 residue classes modulo 2013. By the pigeonhole principle two distinct partial sums must be in the same residue class.

3. Prove or disprove: there exists a function f with domain and range $\mathbb{R}$ such that the equation $f(x) = x$ has exactly one distinct solution and the equation $f(f(x)) = x$ has exactly two distinct solutions.

   Solution: Let A be the set of a so that $f(a) = a$ and let B be the set of b such that $f(f(b)) = b$. Then $A \subseteq B$. Let $B' = B \setminus A$. We claim that if B' is finite, then it has even cardinality—if $b \in B'$, then $f(b)$ is also in B' and is not equal to b. Thus B' is partitioned into pairs $\{b, f(b)\}$. To finish, A is assumed to have cardinality 1, so B, if finite, has odd cardinality.

   Alternate Solution: Suppose such a function exists. Let a be the unique solution to $f(x) = x$ (that is, $f(a) = a$). Then $f(f(a)) = f(a) = a$, so a is also a solution to $f(f(x)) = x$. Let the other distinct solution to this second equation be $b \neq a$. Then, $f(f(f(b))) = f(b)$, so $f(b)$ is also a solution to $f(f(x)) = x$. Since $f(f(x)) = x$ has only two distinct solutions, we must have either $f(b) = b$ or $f(b) = a$. If $f(b) = b$, then b is a second distinct solution to $f(x) = x$, a contradiction with our assumption.

If $f(b) = a$, then $f(f(b)) = a$, so $b = a$, also a contradiction. Therefore, no such function f exists.

Remark: Note that we did not need that the domain and range of f was $\mathbb{R}$. As long as f is a function mapping some set X to itself, the proof would still hold.

4. Let $\triangle ABC$ be a triangle such that $\angle B$ has twice the measure of $\angle A$. Prove that $BC(BC + AB) = AC^2$.

Solution: Let X be a point on AC such that $\overline{BX}$ bisects $\angle B$. Then $\triangle ABC$ and $\triangle BXC$ are similar. Moreover, $\triangle AXB$ is isosceles, so $AX = BX$. Now, since $AC = AX + CX = BX + CX$, use similarity of $\triangle ABC$ and $\triangle BXC$ to get

$$AC = \frac{AB \cdot BC}{AC} + \frac{BC^2}{AC},$$

equivalent to the desired result.

5. For $k \geq 0$, define

$$f(k) = \sum_{n=k}^{\infty} \binom{n}{k} 2^{-n}.$$

Give a closed-form expression for $f(k)$.

Solution: By the identity $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ (i.e., the recursive formula for computing Pascal's triangle), we can write

$$\begin{aligned}
f(k) &= \sum_{n=k}^{\infty} \binom{n-1}{k-1} 2^{-n} + \sum_{n=k}^{\infty} \binom{n-1}{k} 2^{-n} \\
&= \frac{1}{2} \sum_{n=k}^{\infty} \binom{n-1}{k-1} 2^{-n+1} + \frac{1}{2} \sum_{n=k}^{\infty} \binom{n-1}{k} 2^{-n+1} \\
&= \frac{1}{2} \sum_{n=k-1}^{\infty} \binom{n}{k-1} 2^{-n} + \frac{1}{2} \sum_{n=k-1}^{\infty} \binom{n}{k} 2^{-n} \\
&= \frac{1}{2} f(k-1) + \frac{1}{2} f(k) + \binom{k-1}{k} 2^{-n-1} \\
&= \frac{1}{2} f(k-1) + \frac{1}{2} f(k),
\end{aligned}$$

since $\binom{n}{k} = 0$ when $n < k$. Thus,

$$f(k) = f(k-1),$$

so f is constant with respect to k. We can then compute

$$f(0) = \sum_{n=0}^{\infty} \binom{n}{0} 2^{-n} = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = 2,$$

so $f(k) = 2$ for all $k \geq 0$.

Alternate Solution: Recall that the geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Taking $x = 1/2$ gives $f(0)$ on the left-hand side and 2 on the right-hand side. Now, differentiate both sides with respect to x and then multiply each side by x. This gives

$$\sum_{n=1}^{\infty} nx^n = x(1-x)^{-2}.$$

Again, taking $x = 1/2$ gives $f(1)$ on the left-hand side and 2 on the right-hand side. In general, taking k derivatives of the power series expansion for $1/(1-x)$ and then multiplying each side by x^k gives the equation

$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} x^n = k! x^k (1-x)^{-k-1},$$

or

$$\sum_{n=k}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$$

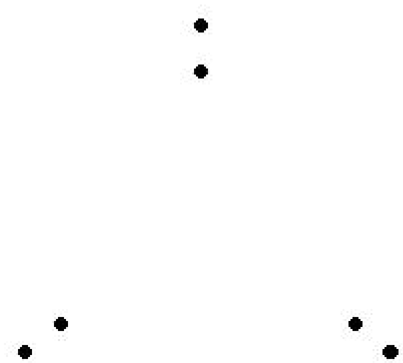
Plugging in $x = 1/2$ then gives $f(k)$ on the left-hand side and 2 on the right-hand side. Thus, $f(k) = 2$ for all k.

6. (Tiebreaker) Let X be a set of points in the plane, no three of which are collinear.
   (a) If $|X| = 5$, show that there is a convex quadrilateral whose vertices are in X.
   (b) If $|X| = 6$, show that there are at least three convex quadrilaterals whose vertices are in X.
   (c) Give an example of a set X with six points and *exactly* three convex quadrilaterals.

Solution: When $|X| = 5$, consider the convex hull of the point set. If the convex hull is a pentagon or quadrilateral, we are done; otherwise the convex hull must be a triangle. Name the points on the hull A, B, C and the points inside the hull D, E. Extend the line DE. WLOG, this line intersects AB and AC, and it is easy to see B, C, D, E form a convex quadrilateral.

When $|X| = 6$, name the six points A, B, C, D, E, F. Consider $S = \{A, B, C, D, E\}$. From the previous case, there exists at least one convex quadrilateral Q_1 within S. WLOG we may assume the vertices of Q_1 are A, B, C, D. Next we consider $T = \{B, C, D, E, F\}$; again by the previous case there is another convex quadrilateral Q_2 within T. Since there must be one common vertex between Q_1 and Q_2, say B, we can apply the lemma to $\{A, C, D, E, F\}$ to get a distinct third convex quadrilateral Q_3. Hence we can always find at least three convex quadrilaterals within six points.

The following figure shows an example of a set X with exactly three convex quadrilaterals.



Problems contributed by Meredith Casey, Santhosh Karnik, Robert Krone, Chris Pryby, and Chi-Ho Yuen. Thanks also to Tobias Hurth and Prof. Doron Lubinsky for assistance in editing.