1. A square and a triangle have equal perimeters. If the lengths of the three sides of the triangle are 4.2, 5.1, and 6.7, what is the area of the square?

   (a) 4

   (b) 8

   (c) 16

   (d) 36

   (e) 25

> **Solution:** C. The perimeter of the triangle is $4.2 + 5.1 + 6.7 = 16$. Since the square has the same perimeter, its side length must be 4. Thus, the square's area is $4 \cdot 4 = 16$.

2. If $a \otimes b = \dfrac{ab}{a-b}$, then what is the value of $2 \otimes (6 \otimes 3)$?

   (a) -3

   (b) $-\frac{3}{2}$

   (c) $\frac{3}{2}$

   (d) 3

   (e) Undefined

> **Solution:** A. First compute $6 \otimes 3 = \frac{18}{3} = 6$. Then compute $2 \otimes 6 = \frac{12}{-4} = -3$.

3. There are three dartboards, each divided into three regions with equal area. The first board's regions are worth 1, 2, and 3 points; the second board's regions are worth 2, 4, and 6 points; and the third dartboard's regions are worth 1, 3, and 5 points.

   If Robert throws one dart at each dartboard, what is the probability that his total score will be even? (Assume each point on a dartboard has an equal probability of being hit.)

   (a) $\frac{5}{27}$

   (b) $\frac{2}{9}$

   (c) $\frac{1}{3}$

   (d) $\frac{5}{9}$

   (e) $\frac{2}{3}$

Solution: E. Since adding an even number doesn't change the parity of a sum, we may ignore the second dartboard. The third dart always lands on an odd number, so we only need the probability that the first dart will land on an odd number (in which case Robert's sum will be even). This probability is $\frac{2}{3}$.

4. In how many ways can eight people be seated in a circle if two arrangements are considered the same whenever each person has the same neighbors (not necessarily on the same side)?

   (a) 1400

   (b) 2520

   (c) 2800

   (d) 5040

   (e) 20160

Solution: B. There are $(n-1)!$ distinct ways of seating n people at a circular table, where two arrangements are considered the same if one is obtained by rotating positions at the table to get to the other (and vice versa): to see this, imagine fixing one person's seat and then permuting the remaining $n-1$ persons' seats.

By not distinguishing between two arrangements if their neighbors are on different sides, we reduce the number of distinct arrangements by a factor of 2: to switch one person's neighbors and still retain the property that everyone's neighbors are the same, we must reverse the order of the cycle.

Thus, in this situation, we have $7!/2 = 2520$ possible arrangements.

5. In front of you are three boxes: one contains just apples, one contains just oranges, and one contains both apples and oranges. However, the three boxes are all mislabeled: the one labeled "Just Apples" is not the box containing just apples, the one labeled "Just Oranges" is not the box containing just oranges, and the one labeled "Both Apples & Oranges" is not the box containing both apples and oranges.

   You are allowed to randomly pick a single fruit from a single box. Based on that fruit, you must rearrange the labels on the boxes to reflect their true contents. Which box should you pick a fruit from in order to relabel the boxes correctly?

   (a) The box labeled "Just Apples"

   (b) The box labeled "Just Oranges"

(c) The box labeled "Both Apples & Oranges"

(d) No matter which box you pick, you can correctly relabel the boxes

(e) No matter which box you pick, you will not have enough information to correctly relabel the boxes

Solution: C. Assume the first box is labeled "Just Apples," the second box is labeled "Just Oranges," and the third box is labeled "Both Apples & Oranges." Since none of the original labels are correct, the only possible box orders are (O, B, A) and (B, A, O), where A stands for the box with just apples inside, O stands for the box with just oranges inside, and B stands for the box with both apples and oranges.

Picking from the first box may not give you enough information, since seeing an orange will not tell you whether that box should be labeled "Just Oranges" or "Both Apples & Oranges." Similarly, picking from the second box gives you no information if you draw an apple from it. However, if you pick from the third box and see an apple, then you know that box must contain only apples; if you see an orange, it must contain only oranges. From this information, you can determine the contents of the remaining two boxes (because there is only one valid ordering in which the third box has only apples and likewise if the third box has only oranges).

6. Let ABC be a right triangle with right angle at B. Draw the segment $\overline{BD}$ so that it bisects $\overline{AC}$. If $m\angle BAD = 52°$, what is $m\angle BDC$?

   (a) 104°

   (b) 38°

   (c) 128°

   (d) 26°

   (e) Cannot be determined from the information given

Solution: A. Circumscribing a circle about ABC, $\overline{AC}$ is a diameter of the circle, so $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$ are all radii of the circle. Since $m\angle BCA = 90° - 52° = 38°$, we also have $m\angle DBC = 38°$; therefore $m\angle BDC = 180° - 76° = 104°$.

7. Brittany is mixing cake batter in a industrial-sized stand mixer. There are eight packages in front of her, seven of which contain flour and one of which contains baking soda. Though the packages are unlabeled, she knows that a package of baking soda is heavier

than a package of flour, and she has a balance scale she can use to determine which of two sets of packages is heavier.

What is the fewest number of weighings she needs in order to determine which package has baking soda?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 7

Solution: B. Put three packages on each side of the balance scale. If their weights are equal, then weigh the remaining two packages to determine which one has the baking soda. If the two sets of three packages are of different weight, then weigh two packages from the heavier side. If they have unequal weight, the heavier one has the baking soda. If they have equal weight, the third package must have the baking soda.

8. Three identical coins of radius 1 are placed on a table so that they are mutually tangent. A smaller coin is placed between them tangent to all 3. What is the radius of the smaller coin?

(a) $\frac{1}{3}$

(b) $\frac{2}{\sqrt{3}} - 1$

(c) $\sqrt{2} - 1$

(d) $\frac{1}{2\sqrt{3}}$

(e) $\sqrt{3} - 1$

Solution: B. Connecting the centers of the three large coins produces an equilateral triangle with side length 2. The center of the triangle is distance $2/\sqrt{3}$ from each of the vertices. This distance is the sum of the radii of a large coin and the small coin, so the small radius is $2/\sqrt{3} - 1$.

9. Denote by $s(N)$ the sum of the decimal digits of N. Find the minimum possible value of $s(66k)$ over all $k \geq 1$.

(a) 2

(b) 3

(c) 6

(d) 9

(e) 12

Solution: C. $66 \cdot 2 = 132$, so the answer is at most 6. $66k$ is a multiple of 11, so the difference between the sum A of odd digits (last, third to last, fifth to last, etc.) and the sum B of remaining digits is a multiple of 11. Now either A or B is ≥ 11, which we may ignore since then $s(66k) \geq 11 > 6$ (the upper bound we have already established); or else $A = B$ and $s(66k)$ is even. Since $66k$ is also a multiple of 3, $s(66k)$ must be a multiple of 6, hence the answer is at least 6.

10. Suppose the parabola $y = ax^2 + bx + c$ passes through the points $(-4, 12)$, $(-2, 0)$, and $(2, 12)$. Find $a + b + c$.

(a) 4.5

(b) -4.5

(c) 10.5

(d) -10.5

(e) None of the above

Solution: A. Plugging in each point to the equation, we obtain the linear system

$$12 = 16a - 4b + c$$
$$0 = 4a - 2b + c$$
$$12 = 4a + 2b + c$$

This system has the solution $a = 1.5, b = 3, c = 0$, so the answer is 4.5.

11. How many pairs of integers (x, y) with $1 \leq x \leq 100$ and $1 \leq y \leq 100$ are there such that $2x^2 + 3y^2$ is divisible by 5?

(a) 10000

(b) 400

(c) 1600

(d) 2000

(e) 3600

Solution: E. The pair of remainders of (x, y) modulo 5 should be $(0, 0)$, $(\pm 1, \pm 1)$, or $(\pm 2, \pm 2)$. There are $20 \cdot 20 = 400$ pairs of the first kind and $(20 \cdot 2)^2 = 1600$ pairs of the second kind and third kind (each). $400 + 1600 + 1600 = 3600$.

12. A cockroach is in the center of a square trash compactor with side length 1 meter. Two opposing walls begin contracting towards the center of the trash compactor at a rate of $\frac{1}{10}$ meter per minute until they meet, squishing the cockroach. Sensing its impending doom, the cockroach runs back and forth between the two contracting walls. If the cockroach has a constant speed of 80 meters per minute, what is the total distance (in meters) the cockroach will cover before it is squished?

    (a) 800
    (b) 400
    (c) $200\sqrt{2}$
    (d) $400\sqrt{2}$
    (e) $200\sqrt{3}$

Solution: B. The distance from each side to the center of the trash compactor is 0.5 m. Since the walls are moving in at a speed of 0.1 m/min, the walls will squish the cockroach in $\frac{0.5 \text{ m}}{0.1 \text{ m/min}} = 5$ min. Since the cockroach is moving at a constant speed of 80 m/min, in 5 min, the cockroach can cover a total distance of (80 m/min)(5 min) = 400 m.

13. What is the remainder when $(x - 2)^{2013}$ is divided by $(x - 1)$?

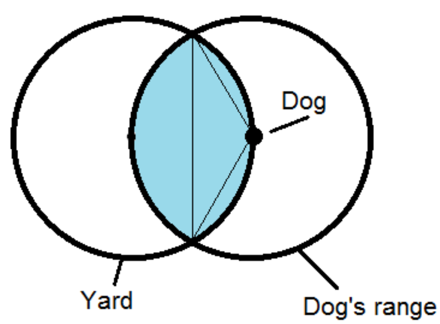
    (a) -1
    (b) 1
    (c) 2
    (d) 2013
    (e) 2^{2013}

Solution: A. By the polynomial remainder theorem, the remainder when a polynomial $f(x)$ is divided by a linear factor $x - a$ is $f(a)$. Thus, the solution is $(-1)^{2013} = -1$.

14. A circular yard has radius 1. A dog is tethered to a point on the edge of the yard by a leash of length 1. What area of the yard can the dog reach?

    (a) $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$
    (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$
    (c) $\frac{2\pi}{3} - \sqrt{3}$
    (d) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$
    (e) $\frac{2\pi}{3} - \frac{\sqrt{2}}{2}$

Solution: A. Consider the diagram below. We are given two unit circles, one representing the yard, and one representing the range the dog can reach while tethered. The shaded area is the portion of the yard reachable by the dog. If we take the center of the yard to be the point $(0, 0)$ and the base of the dog's leash to be the point $(1, 0)$, then the vertical chord above occurs along the line $x = \frac{1}{2}$. Thus, the intersection points of the circles are at $(\frac{1}{2}, \pm\frac{\sqrt{3}}{2})$. The angle subtended by the isosceles triangle above is then $120°$. The left half of the shaded region plus the isosceles triangle make up one-third the area of the right circle, which is $\frac{\pi}{3}$. Taking away the area of the triangle, which is $\frac{\sqrt{3}}{4}$, we see that the left half of the shaded region has area $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$. Since this is half of the area we wanted, the final answer is therefore $\frac{2\pi}{3} - \frac{\sqrt{3}}{2}$.



15. The sum of the squares of three prime numbers is 10214. What is the sum of the three prime numbers?

    (a) 104
    (b) 106
    (c) 141
    (d) 170
    (e) 172

Solution: B. Let the primes be p, q, r; then $p^2 + q^2 + r^2 = 10214$. Since $p^2 + q^2 + r^2$ is even, at least one of p^2, q^2, r^2 is even, so at least one of p, q, r is even (i.e., 2). Let's say that $r = 2$; then $p^2 + q^2 = 10210$. Every perfect square is either 0 or 1 (mod 3). Since $p^2 + q^2 = 10210 \equiv 1 \pmod 3$, we have that one of p^2, q^2 is $0 \pmod 3$. Thus, one of p, q is congruent to $0 \pmod 3$. The only prime divisible by 3 is 3 itself, so one of the primes (let's say q) equals 3. This leaves $p^2 = 10201$, so $p = 101$ (which is prime). Therefore, $p + q + r = 101 + 3 + 2 = 106$.

16. In a certain computer game, two players each start with 0 points, and they play a series of rounds resulting in exactly one of the players gaining one more point (a round cannot end in a tie). The game ends when one player gains 21 points, and that player wins the game.

The players' scores during a game are represented as the ordered pair (x, y). What is the fewest number of games that must be played in order for all possible pairs of scores to show up at least once? [Remark: There are $22 \cdot 22 - 1 = 483$ possible pairs of scores total.]

(a) 12
(b) 21
(c) 22
(d) 42
(e) 84

Solution: D. In a single game, at most one score pair where one of the scores is 21 can show up; since there are 42 such score pairs, we must play at least 42 games in order to see all the pairs at least once.

Now we construct a set of 42 games such that every possible score pair appears at some point in some game. For the i^{th} game (where $i = 1, ..., 21$), Player 1 wins the first $i - 1$ rounds, and Player 2 wins the remaining 21 rounds. For the $(21 + i)^{\text{th}}$ game (where $i = 1, ..., 21$), Player 2 wins the first $i - 1$ rounds, and Player 1 wins the remaining 21 rounds.

17. A positive integer is *fuzzy* if every two consecutive digits (when written in base 10) forms a number that is a multiple of either 19 or 21. For example, the number 57638 is fuzzy because 57, 76, 63, and 38 are all multiples of 19 or 21; however, the number 3838 is not fuzzy because 83 is not a multiple of either 19 or 21.

How many 2013-digit numbers (when written in base 10) are fuzzy?

(a) 9
(b) 10
(c) 19
(d) 20
(e) 21

Solution: A. The two-digit multiples of 19 are $19, 38, 57, 76, 95$. The two-digit multiples of 21 are $21, 42, 63, 84$. From this, it is clear that the digit 0 cannot appear in a fuzzy 2013-digit number. Also, if the n^{th} digit in a fuzzy number is $1, 2, 3, 4, 5, 6, 7, 8, 9$, then the $(n+1)^{\text{th}}$ digit must be a $9, 1, 8, 2, 7, 3, 6, 4, 5$ respectively. Thus, after choosing the first digit between 1 and 9, the remaining 2012 digits are fixed. Hence, there are 9 fuzzy 2013-digit numbers.

18. A cube is inscribed in a sphere which is inscribed in a cube which is inscribed in a sphere which is inscribed in a cube. Find the ratio of the volume of the outermost cube to the volume of the innermost cube.

(a) 3
(b) 4
(c) 8
(d) 9
(e) 27

Solution: E. Let the innermost cube have vertices $(\pm 1, \pm 1, \pm 1)$, so its side length is 2. The inner sphere has center $(0, 0, 0)$ and touches point $(1, 1, 1)$ so it has radius $\sqrt{3}$. The middle cube then has side length $2\sqrt{3}$, a factor of $\sqrt{3}$ larger. Similarly the outer cube has side $\sqrt{3}$ times larger than the middle cube, so 3 times larger than the small cube. Since the side length is 3 times larger, the volume is $3^3 = 27$ times larger.

19. How many scalene triangles are there whose vertices lie on the vertices of a cube?

(a) 8
(b) 12
(c) 24
(d) 32

(e) 48

Solution: C. There are $\binom{8}{3} = 56$ possible triangles to be made by choosing three points from the eight vertices of a cube. We can make an isosceles triangle in two ways: by choosing two opposing vertices on a face of the cube and then choosing one of the other vertices on the same face (this gives four triangles per face, which is 24 triangles) or by choosing two opposing vertices on a face and then choosing a third vertex not adjacent to either of these two (there are eight such triangles). Scalene triangles may be made by selecting two vertices sharing an edge and then choosing one of the two vertices not adjacent to either of these two. Since there are 12 edges on the cube, this yields 24 scalene triangles. Since $24 + 8 + 24 = 56$, we have accounted for all the triangles.

20. Evaluate $\displaystyle\sum_{n=0}^{89} \frac{1}{1 + \tan^3 n}$. [Use degrees not radians.]

(a) $\frac{30\pi}{2}$

(b) 50

(c) $\sqrt{2339}$

(d) $30\sqrt{3}$

(e) 45.5

Solution: E. Note that

$$\sum_{n=1}^{89} \frac{1}{1+\tan^3 n} = \sum_{n=1}^{89} \frac{\cos^3 n}{\cos^3 n + \sin^3 n} = \sum_{n=1}^{89} \frac{\sin^3 n}{\sin^3 n + \cos^3 n}.$$

Hence

$$\sum_{n=1}^{89} \frac{1}{1+\tan^3 n} = \frac{1}{2}\left(\sum_{n=1}^{89} \frac{\cos^3 n}{\cos^3 n + \sin^3 n} + \sum_{n=1}^{89} \frac{\sin^3 n}{\sin^3 n + \cos^3 n}\right) =$$

$$\frac{1}{2}\sum_{n=1}^{89} \frac{\cos^3 n + \sin^3 n}{\cos^3 n + \sin^3 n} = \frac{1}{2}\sum_{n=1}^{89} 1 = \frac{89}{2} = 44.5.$$

Adding the case $n = 0$ into the summation, we get 45.5.

21. Evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{3^{n-1}}$.

(a) $\frac{3}{2}$

(b) 2

(c) 3

(d) $\frac{9}{2}$

(e) 6

Solution: D. Let

$$S = \sum_{n=1}^{\infty} \frac{n^2}{3^{n-1}} = \frac{1}{3^0} + \frac{4}{3^1} + \frac{9}{3^2} + \frac{16}{3^3} + \cdots .$$

Multiply by $\frac{1}{3}$ to get

$$\frac{1}{3}S = \frac{1}{3^1} + \frac{4}{3^2} + \frac{9}{3^3} + \frac{16}{3^4} + \cdots .$$

Subtract the last 2 equations to get

$$\frac{2}{3}S = \frac{1}{3^0} + \frac{3}{3^1} + \frac{5}{3^2} + \frac{7}{3^3} + \cdots .$$

Multiply by $\frac{1}{3}$ to get

$$\frac{2}{9}S = \frac{1}{3^1} + \frac{3}{3^2} + \frac{5}{3^3} + \frac{7}{3^4} + \cdots .$$

Subtract the last 2 equations to get

$$\frac{4}{9}S = \frac{1}{3^0} + \frac{2}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots .$$

Adding $1 = \frac{1}{3^0}$ to both sides makes the right side a geometric series:

$$\frac{4}{9}S + 1 = \frac{2}{3^0} + \frac{2}{3^1} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots = \frac{2}{1 - 1/3} = 3.$$

Since $\frac{4}{9}S + 1 = 3$, we have $S = \frac{9}{2}$.

22. A circle is inscribed inside a rhombus. If one diagonal of the rhombus is three times as long as the other and the rhombus has area 24, give the area of the inscribed circle.

(a) $\frac{9\pi}{10}$

(b) 16π

(c) $\frac{360\pi}{25}$

(d) $2\sqrt{2}$

(e) None of the above

Solution: E. The area of a rhombus is half the product of its diagonals; thus, $\frac{3a \cdot a}{2} = 24$, so $a = 4$. The half-diagonals then have length 2 and 6, so the radius of the inscribed circle is

$$r = \frac{2 \cdot 6}{\sqrt{2^2 + 6^2}} = \frac{6}{\sqrt{10}}.$$

Thus the area of the circle is $\pi r^2 = \frac{36}{10}\pi = \frac{18}{5}\pi$.

23. In how many ways can a 6×10 board of unit squares be tiled using rectangular tiles of size 2×3, oriented horizontally and vertically?

(a) 7

(b) 8

(c) 9

(d) 10

(e) 11

Solution: A. See how we cover the bottom-left corner square. It can be covered by a tile horizontally or vertically. Once it is covered by a horizontal one, the we must use a horizontal tile to cover the top-left square. Similarly, if the left-bottom square is covered by a vertical tile, then the top-left square should be covered by a vertical tile as well. So it gives the recursive relation that $T(n) = T(n-2) + T(n-3)$, where $T(n)$ is the number of ways to tile $6 \times n$ squares. Use $T(1) = 0, T(2) = 1, T(3) = 1$ to solve $T(10)$.

Alternatively, we can count the number of ways we can obtain 10 by using the summands 2 and 3. Since $10 = 5 \cdot 2 = 2 \cdot 2 + 2 \cdot 3$, this gives $\frac{5!}{5!} + \frac{4!}{2!2!} = 7$ ways.

24. Suppose 100,000 people stand in a line, all facing towards the front of the line. The person at the front of the line is assigned the number 1, the second person in line is assigned the number 2, and so forth. In each round, a person will leave the line if there is an even number of people standing in front of him or her. Which of these numbers is assigned to a person who will leave in the 14^{th} round?

(a) 10240

(b) 20480

(c) 24576

(d) 29000

(e) 32768

Solution: C. The n^{th} person in the line at beginning will leave the line at the i^{th} round if 2^{i-1} is the smallest power of 2 that evenly divides n. So n must be evenly divisible by $2^{13} = 8192$ but not by $2^{14} = 16384$. So the answer is 24576.

25. The curve $y = x^4 - 8x^3 + 9x^2 + 20x + 2$ and the line $y = 2x + 1$ intersect at four distinct points in the real xy-plane. Find the average y-value of the intersection points.

    (a) -2

    (b) 0

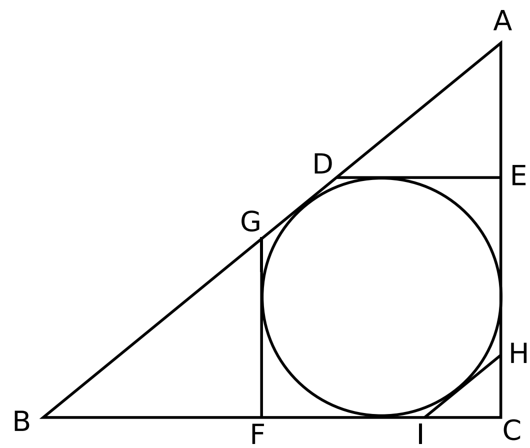
    (c) 2

    (d) 5

    (e) 8

Solution: D. The x-values of the intersection points satisfy $x^4 - 8x^3 + 9x^2 + 20x + 2 = 2x + 1$, so they are the roots of the quartic $x^4 - 8x^3 + 9x^2 + 18x + 1$. The x^3-coefficient shows that the sum of the roots is 8, so the average x-value is $\overline{x} = 2$. Since all the intersection points lie on the line $y = 2x + 1$, the average y-value is $\overline{y} = 2\overline{x} + 1 = 5$.

26. Triangle ABC has side lengths $3, 4, 5$ for sides AC, BC, and AB respectively. A circle is inscribed in the triangle and then segments DE, FG, and HI are drawn so that they are parallel to BC, AC, and AB respectively, and also tangent to the inscribed circle. Compute the sum of the perimeters of the three small triangles formed: ADE, BFG, and CHI.



(a) 7

(b) 9

(c) 12

(d) $\frac{25}{2}$

(e) 15

Solution: C. Note that the three new tangent lines form a triangle congruent to ABC (rotated 180°). Using these resulting congruences, the sum of the perimeters of the small triangles is equal to the perimeter of ABC, which is 12

27. Suppose you have a bucket with 3 balls, labeled A, B, and C. You draw a ball from the bucket at random and then return it, repeating this process until you have seen each of the three balls at least once. Find the average number of draws needed to see all three balls.

(a) 4

(b) $\frac{11}{2}$

(c) $\frac{20}{3}$

(d) 9

(e) ∞

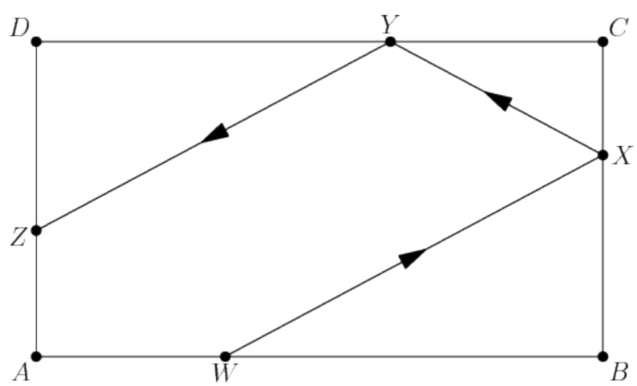
Solution: B. (This is the "Coupon Collector's Problem.") Suppose you're at the stage of having already seen k distinct balls out of the 3. The probability of drawing a number you've seen is $\frac{k}{3}$ and the probability of drawing a new ball is $\frac{3-k}{3}$. The expected number of draws needed to find a new ball x satisfies the equation

$$x = 1 + \frac{k}{3} \cdot x + \frac{3-k}{3} \cdot 0.$$

This is because you have to make 1 draw no matter what, and depending on the result you are back where you started and need another x expected draws, or you've reached your goal of seeing a new number and need no more draws. Solving for x produces $x = \frac{3}{3-k}$. Then total expected draws T is the sum of the draws needed for each k, so

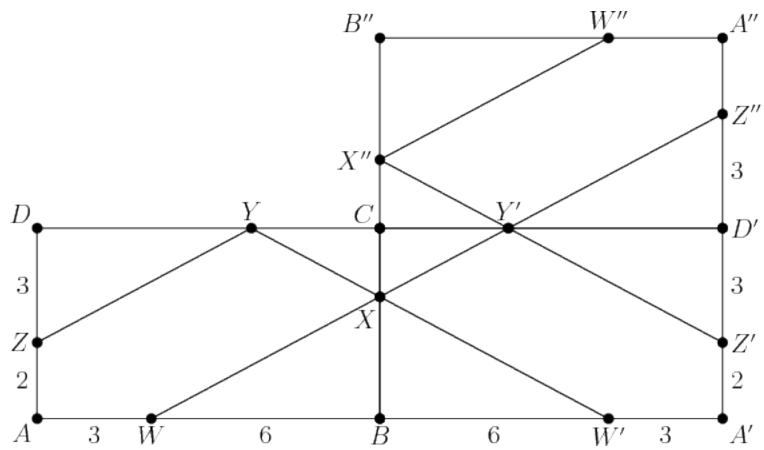
$$T = \frac{3}{3} + \frac{3}{2} + \frac{3}{1} = \frac{11}{2}.$$

28. A rectangular billiard table $ABCD$ shown below has dimensions $AB = 9$ and $AD = 5$. Points W and Z are on AB and AD respectively such that $AW = 3$ and $AZ = 2$. A ball starts at W, bounces off BC, then bounces off CD, and finally hits AD at point Z and stops. Find the total distance traveled by the ball.



(a) 15
(b) 16
(c) 17
(d) 18
(e) 19

Solution: C. From the given information, we have $BW = 6$ and $DZ = 3$. Now, reflect $ABCD$ about side BC to get $A'B'C'D'$. Then reflect $A'B'C'D'$ about side $C'D'$ to get $A''B''C''D''$. Note that W, $X = X'$, $Y = Y''$ and Z'' are collinear. Thus, the total distance traveled by the ball is $WX + XY + YZ = WX + X'Y' + Y''Z'' = WZ''$. Since $WA'Z''$ is a right triangle with legs $WA' = 15$ and $A'Z'' = 8$, the hypotenuse has length $WZ'' = \sqrt{15^2 + 8^2} = 17$.



29. Chris lives at $(0, 0)$ on an integer street grid where all streets run east or north. The beach is along the line $x + 2y = 12$. How many different ways can Chris drive to the beach? (Assume that he must end at an integer point on the shoreline—he doesn't want to drive into the ocean!)

    (a) 66
    (b) 132
    (c) 144
    (d) 180
    (e) 233

Solution: E. Define the Fibonacci sequence by $a_0 = 1$, $a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Then the answer is $a_{12} = 233$. To see this, observe that the number $P(x, y)$ of taxi paths from the origin to (x, y) is $\binom{x+y}{x}$ (there are $x + y$ line segments in the path, and exactly x must be chosen to be horizontal); also, the set of integer points (x, y) satisfying

$$x + 2y = 2d$$

are points of the form $(x, d - \frac{x}{2})$, where x is an even integer varying from 0 to $2d$. Thus,

$$\sum_{\substack{x \geq 0, y \geq 0 \\ x+2y=2d}} P(x, y) = \sum_{k=0}^{d} \binom{d+k}{k}.$$

This sum is computable by hand for small values of d, but it is easy to show by an induction argument that this sum in fact equals a_{2d}.

30. Evaluate $\sum_{k=1}^{360} \lceil 6 \sin k \rceil$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. [Use degrees not radians.]

    (a) 0
    (b) 6
    (c) 176
    (d) 179
    (e) 180

Solution: C. Since $\sin k = -\sin(k + 180)$ we can regroup the sum as

$$\sum_{k=1}^{180} \lceil 6 \sin k \rceil + \lceil -6 \sin k \rceil.$$

Note that $\lceil x \rceil + \lceil -x \rceil$ is equal to 0 if x is an integer and 1 otherwise. For k from 1 to 180, the expression $6 \sin k$ is an integer only for $k = 30, 90, 150, 180$. The other 176 values of k each contribute 1 to the sum.

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