Georgia Tech High School Math Competition

Team Test — Solutions

March 3, 2012

Write your names and answers on the provided answer sheet. You may collaborate only within your group on this test. This test will last 90 minutes.

1. A regular tetrahedron is inside a right cone in such a way that the tip of the cone is at one vertex of the tetrahedron and the other three vertices lie on the circle at the base of the cone. If each side of the tetrahedron has length 1, find the volume of the region outside the tetrahedron but inside the cone.

Solution: The side lengths of the tetrahedron are length 1 and the volume of a regular tetrahedron with side length a is $\frac{\sqrt{2}a^3}{12}$, so the volume of the tetrahedron is $\sqrt{2}/12$. The volume of a right cone with circular base is $\pi r^2 h/3$. The height of the cone equals the height of the tetrahedron, which is $\sqrt{6}/3$. The radius is found by using a 30-60-90 triangle on the base, giving the radius as $\sqrt{3}/3$. Plugging in we get the volume of the cone to be $\frac{\sqrt{2}\pi}{9\sqrt{3}}$. From there we just subtract and simplify to obtain

$$\frac{\sqrt{2}\pi}{9\sqrt{3}} - \frac{\sqrt{2}}{12} = \frac{4\sqrt{6}\pi - 9\sqrt{2}}{108}$$

Note that the height and volume of a tetrahedron can be derived geometrically without prior knowledge of the formulas.

2. Suppose you have a box containing five balls, marked A, B, C, D, and E. You start drawing balls from the box randomly, replacing the ball you drew each time before drawing again. You stop drawing after you see all five balls. What is the expected number of times you will draw a ball?

Solution: $1 + 5/4 + 5/3 + 5/2 + 5 = 137/12$

3. A regular tetrahedron $ABCD$ has edge length 1. Find the shortest distance between edge AB and edge CD.

> **Solution:** The shortest distance is from the midpoint E of AB to the midpoint of CD. Triangle ABC is equilateral, so the length of EC is $\sqrt{3}/2$, and similarly the length of ED. Triangle ECD is isosceles with known side lengths, so we can compute its height, $\sqrt{2}/2$, which is the answer.

4. A circular dart board is hung on a wall by a nail at the top edge of the board. You throw a dart and hit the board. Assuming you have a uniform probability of hitting any spot on the dart board, what is the probability that the dart lands closer to the nail than to the center of the board?

> **Solution:** Assume the board has radius 1. The points closer to the nail are those above the line halfway between the center and the nail. This chord cuts across an arc of length $\pi/3$, so the area above the line is the slice of the circle of area $\pi/3$ minus the triangle below the line. The triangle below the line has area $\sqrt{3}/4$. The total area of the board is π, so the probability of being above the line is $\frac{1}{3} - \frac{\sqrt{3}}{4}$.

5. How many consecutive zeros are there at the end of 2012! (in base ten)?

> **Solution:** The number of zeros at the end of 2012! equals the number of factors of five in 2012! (since there will be more factors of two than factors of five). In $1, 2, ..., 2012$ there are $\lfloor 2012/5 \rfloor = 402$ numbers divisible by 5, $\lfloor 2012/25 \rfloor = 80$ numbers divisible by 25, 16 divisible by 125, and 3 divisible by 625. Hence we have $402 + 80 + 16 + 3 = 501$ factors of five, and therefore 501 zeros at the end of 2012!.

6. Find all positive integers n for which $n(n + 9)$ is a perfect square.

> **Solution:** It is easy to check that $n^2 < n(n + 9) < (n + 5)^2$, so $n(n + 9)$ should be equal to one of $(n + 1)^2$, $(n + 2)^2$, $(n + 3)^2$, or $(n + 4)^2$. For $n(n + 9) = (n + 1)^2$ we get $7n = 1$, which has no integer solutions. For $n(n + 9) = (n + 2)^2$ we get $5n = 4$, which has no integer solutions. For $n(n + 9) = (n + 3)^2$ we get $3n = 9$, so $n = 3$. For $n(n + 9) = (n + 4)^2$ we get $n = 16$.

7. Suppose $\{v_1, ..., v_6\}$ is a set of six vectors in the plane each with magnitude at least 1. Find the maximum number of *unordered* pairs of vectors $\{v_i, v_j\}$ such that $||v_i + v_j|| < 1$.

Solution: Let's say $v_i \sim v_j$ if $||v_i + v_j|| < 1$. Observe that if $v_i \sim v_j$ and $v_j \sim v_k$, then $v_i \not\sim v_k$. To see why, if $v_i \sim v_k$, then v_k must lie in the open disk of radius 1 centered about $-v_i$, which we'll call $B_1(-v_i)$. Moreover, v_k must also lie in the open ball of radius 1 about $-v$ for some $v \in B_1(-v_i)$, so

$$v_k \in \bigcup_{v \in B_1(-v_i)} B_1(-v) = B_2(v_i).$$

However, since $||v_i|| \geq 1$, $B_2(v_i)$ and $B_1(-v_i)$ are disjoint.

Since no three vectors are mutually related by the relation $\sim$, we can rewrite the problem as: letting G be a graph whose vertices are $\{1, ..., 6\}$ and whose edges are between vertices i, j such that $v_i \sim v_j$, what is the maximum number of edges G can have without having any triangles? The maximum is 9, which is attained by the complete bipartite graph $K_{3,3}$. (By Mantel's theorem, the maximum number of edges for a triangle-free graph on n edges is $\lfloor n^2/4 \rfloor$.)