1. Prove that the product of four consecutive positive integers cannot be a perfect square.

Solution: Let $n \geq 1$, and consider

$$n(n+1)(n+2)(n+3) = n(n+3) \cdot (n+1)(n+2) = (n^2+3n)(n^2+3n+2).$$

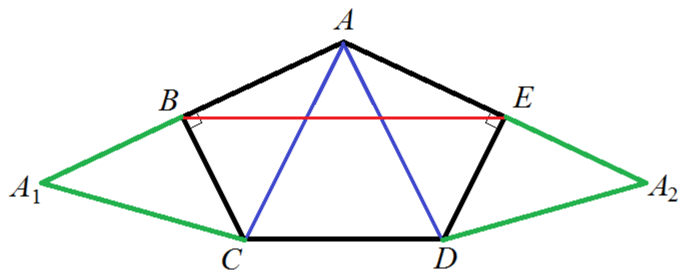
Rewrite the factors as

$$\big((n^2+3n+1)-1\big)\big((n^2+3n+1)+1\big) = (n^2+3n+1)^2 - 1,$$

which is one less than a perfect square. Since no two perfect squares differ by one (except 0 and 1, but since $n \geq 1$ we cannot have $(n^2+3n+1)^2 = 1$), we conclude that $n(n+1)(n+2)(n+3)$ is not a perfect square.

2. Let $ABCDE$ be a convex pentagon such that the angles at vertices B and E are both $90°$. Show that the perimeter of the triangle ACD is at least twice the length of the segment BE.

Solution: Consider the right triangles ABC and AED. Reflect ABC about the segment BC (and AED about ED) to form two isosceles triangles AA_1C and AA_2D (where the length of the segment AB equals the length of A_1B, and the length of AE equals that of A_2E). The perimeter of ACD is then equal to the length of the zigzag A_1CDA_2, which is at least the length of the line segment A_1A_2. Finally, since B is the midpoint of AA_1 and E is the midpoint of AA_2, it follows that the segment A_1A_2 has twice the length of BE (via similar triangles).



3. If A is a non-empty subset of the integers and $\phi : A \to A$ is a function, we say that ϕ is *special* if

   1. $\phi(x)$ is an odd integer whenever $x \in A$ is an even integer, and
   2. ϕ is *onto*, that is, for each $y \in A$, there exists some $x \in A$ such that $\phi(x) = y$.

   Let $\mathcal{S}(A)$ denote the number of special functions $\phi : A \to A$. Determine, with proof, the number of nonempty subsets $A \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ satisfying $1 \leq \mathcal{S}(A) \leq 100$.[1]

Solution: First observe that, for finite sets A, $\phi : A \to A$ is onto if and only if ϕ is one-to-one (that is, distinct elements have distinct preimages). Let $A \subseteq \mathbb{Z}$ be a nonempty, finite set with e even elements and o odd elements. If $e > o$, then no function $\phi : A \to A$ can be one-to-one (hence no functions $\phi : A \to A$ are onto, hence no such ϕ is special) because ϕ would map e even elements to $o < e$ odd elements, so at least two even elements must map to the same odd element.

Now suppose $e \leq o$. For the sake of clarity, denote the domain of ϕ by D and the range of ϕ by R (so D and R have the same elements as A but are separate sets). We can construct a special function $A \to A$ by assigning an odd element of R to each even element of D and then arbitrarily assigning the remaining elements of R to the odd elements of D. There are $\frac{o!}{(o-e)!}$ ways of initially assigning odd elements of R to even elements of D, and then we have $o!$ ways of assigning the remaining o elements of R to the odd elements of D. Thus, $\mathcal{S}(A) = \frac{o!}{(o-e)!} \cdot o! = \frac{(o!)^2}{(o-e)!}$.

We must therefore count subsets $A \subseteq \{1, ..., 8\}$ such that $o + e \geq 1$, $0 \leq e \leq o \leq 4$, and $\frac{(o!)^2}{(o-e)!} \leq 100$. The only 11 integer pairs (o, e) satisfying these conditions are

$$(1,0); (1,1); (2,0); (2,1); (2,2); (3,0); (3,1); (3,2); (3,3); (4,0); \text{ and } (4,1).$$

The set $\{1, ..., 8\}$ has 4 even and 4 odd elements, so we count the number of subsets of $\{1, ..., 8\}$ satisfying one of these 11 requirements.

$$\binom{4}{1}\binom{4}{0} + \binom{4}{1}\binom{4}{1} + \binom{4}{2}\binom{4}{0} + \binom{4}{2}\binom{4}{1} + \binom{4}{2}\binom{4}{2} + \binom{4}{3}\binom{4}{0} + \binom{4}{3}\binom{4}{1} + \binom{4}{3}\binom{4}{2} + \binom{4}{3}\binom{4}{3} + \binom{4}{4}\binom{4}{0} + \binom{4}{4}\binom{4}{1} =$$

$$4 + 16 + 6 + 24 + 36 + 4 + 16 + 24 + 16 + 1 + 4 = 151.$$

[1] The problem, as originally written, asked for the number of nonempty subsets satisfying $\mathcal{S}(A) \leq 100$, which allowed for sets A satisfying $\mathcal{S}(A) = 0$. The answer in this case would be 244. Correct proofs arriving at either this result or the intended result were given full credit.

4. A cone is formed by lines passing through the point $(0, 0, 1)$ and the circle $\{(x, y, 0) : (x-1)^2 + y^2 = 1\}$. If a plane $z = \alpha x + \beta$ with $\alpha > 0$ and $\beta \geq 0$ intersects the cone in a circle as indicated in the figure, then what is α? (Figure not to scale.)

Solution: The cone is determined by the relation $x^2 - 2x(1-z) + y^2 = 0$. Since parallel planes intersect a cone in geometrically similar figures, we may assume $\beta = 0$. (Note the information that $\beta > 0$ is irrelevant.) Replacing $z = \alpha x$ in the relation above, we see that the projection of the circle onto the xy-plane is the ellipse

$$(1+2\alpha)\left(x - \frac{1}{1+2\alpha}\right)^2 + y^2 = \frac{1}{1+2\alpha}.$$

On the other hand, the intersection of the line $z = \alpha x$, $y = 0$ and the line $z = -x/2 + 1$, $y = 0$ (which lies on the cone) is the point $(2/(1+2\alpha), 0, 2\alpha/(1+2\alpha))$. Since the segment connecting this point to the origin is a diameter of the tilted intersection circle, we see that the center and radius of the tilted intersection circle are $(1/(1+2\alpha), 0, \alpha/(1+2\alpha))$ and $\sqrt{1+\alpha^2}/(1+2\alpha)$ respectively. It follows that the points (x, y, z) on the circle satisfy

$$\left(x - \frac{1}{1+2\alpha}\right)^2 + y^2 + \left(z - \frac{\alpha}{1+2\alpha}\right)^2 = \frac{1+\alpha^2}{(1+2\alpha)^2}.$$

Eliminating $z = \alpha x$ as before, we get the elliptical projection

$$(1+\alpha^2)\left(x - \frac{1}{1+2\alpha}\right)^2 + y^2 = \frac{1+\alpha^2}{(1+2\alpha)^2}.$$

Since the two ellipses must be the same, we see that $1 + 2\alpha = 1 + \alpha^2$. Thus, there are two plane angles that will give a circular intersection: $\alpha = 0$ which corresponds to the base circle and $\alpha = 2$ which gives the tilted one.

5. Suppose A is a set of points on the circle $x^2 + y^2 = 1$ with the following property: for every subset $B \subseteq A$, there is an n-gon P in the plane such that B is contained in P and so that $A \setminus B$ has no points in P. What is the maximum size A can have?

Solution: Suppose we are given sets A and B as in the problem. Partition the circle into arcs $C_1, ..., C_t$, such that for each $i = 1, ..., t$ either (a) $C_i \cap B = \emptyset$ or (b) $C_i \cap (A \setminus B) = \emptyset$. In fact, take a partition which minimizes t. Note that t is even, since two adjacent arcs satisfying the same condition can be combined into a single arc. If $t \leq 2n$, then there is a natural $(t/2)$-gon which separates the points in A as desired by putting one edge through the endpoints of each arc of type (a) (and thus forcing the corresponding arc to be outside the resulting polygon, while the remaining arcs will be inside). Indeed, any n-gon can only separate the circle into $2n$ arcs (as each edge can intersect the circle at most twice), so this is best possible. Note that if the points can be separated by a k-gon, they can also be separated by a $(k+1)$-gon (by cutting off a corner).

So now we just need to find the largest number of points which cannot be bipartitioned in a way which will force more than $2n$ arcs in the above formulation. Clearly $2n$ points cannot be partitioned to force more than $2n$ such arcs (as we can assume that each arc contains some point in A). Similarly, for $2n+1$ points there must be two adjacent points in the same part, so there are again at most $2n$ arcs. However, $2n+2$ points can be partitioned in an alternating fashion, forcing $2n+1$ arcs, which thus cannot be separated by an n-gon. Thus the answer is $2n+1$.

6. (Tiebreaker question.) Give the shortest self-contained proof you can of the following statement: There are arbitrarily large gaps between consecutive prime numbers.

Solution: Though there are many ways of proving this statement, here is a purely elementary proof: given $k \geq 2$, the sequence $k! + 2, k! + 3, ..., k! + k$ consists of $k - 1$ consecutive composite numbers. Thus, for every k, there exists a gap of at least size k between two consecutive prime numbers.

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