Georgia Tech High School Math Competition

Multiple Choice Test — Solutions

March 3, 2012

1. How many integers $1 \leq n \leq 2012$ are divisible by 9 or 21 but not both?
   (a) 223
   (b) 254
   (c) 256
   (d) 287
   (e) 318

   Solution: C. There are 223 integers in $[1, 2012]$ divisible by 9, and there are 95 divisible by 21. Being divisible by both means that one is divisible by their least common multiple, $3 \cdot 3 \cdot 7 = 63$. There are 31 such integers in $[1, 2012]$. Thus, there are $(223-31)+(95-31) = 256$ integers divisible by just one of 9 and 21.

2. A paper cone has height 12 and the diameter of the base has length 10. The cone is cut along the side and unrolled into a portion of a disk. What angle of the circle does this portion include?
   (a) $5\pi/13$
   (b) $5\pi/12$
   (c) $10\pi/13$
   (d) $5\pi/6$
   (e) $5\pi/3$

   Solution: C. The base of the cone (before being unrolled) has circumference 10π. The radius of the portion of the disk formed after the cone is unrolled is 13 (since the cone's radius is 5 and the height is 12, its slant height is 13). The circumference of a full circle of radius 13 is 26π. Since the unrolled cone covers $10/26$ of the full disk of radius 13, it therefore occupies $10\pi/13$ radians of angular distance.

3. How many integers between 10000 and 99999 (inclusive), when written in base ten, are palindromes?

   (a) 729
   (b) 810
   (c) 900
   (d) 999
   (e) 1000

Solution: C. There are nine possible choices for the first/last digit, ten possible choices for the second/fourth digit, and ten possible choices for the middle digit.

4. How many positive integers divide $(4!)^4$ evenly?

   (a) 32
   (b) 48
   (c) 59
   (d) 64
   (e) 65

Solution: E. $(4!)^4 = (2^3 \cdot 3)^4 = 2^{12} \cdot 3^4$. Any positive integer dividing this number evenly must be of the form $2^a \cdot 3^b$, where $0 \leq a \leq 12$ and $0 \leq b \leq 4$. This gives us $13 \cdot 5 = 65$ choices of divisors.

5. Let a and b be integers whose sum is evenly divisible by three. Which of the following must be true?

   I. $a^2 + b^2$ is evenly divisible by 3
   II. $a^2 - b^2$ is evenly divisible by 3
   III. $a^3 + b^3$ is evenly divisible by 3

   (a) I only
   (b) II only
   (c) I and III
   (d) II and III
   (e) I, II, and III

Solution: D. Statement I has many counterexamples: $a = 1, b = 2$ for example. II is true because $a^2 - b^2 = (a+b)(a-b)$, and III is true because $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$.

6. Suppose the triangle ABC has a right angle at C and that $BC = 3$ and $AC = 4$. What is the length of the angle bisector at B?

   (a) $\frac{3}{2}\sqrt{5}$

   (b) $2\sqrt{3}$

   (c) $\frac{7}{2}$

   (d) 4

   (e) $3\sqrt{2}$

Solution: A. Let D be the point where the angle bisector meets AC. Because BD is an angle bisector, $\frac{AD}{CD} = \frac{AB}{CB}$ by the angle bisector theorem. Therefore $AD = \frac{5}{2}$ and $CD = \frac{3}{2}$. Finally, by the Pythagorean theorem, $BD = \frac{3}{2}\sqrt{5}$.

7. How many roots does the function $f(x) = \sin^4(x) - \cos^4(x)$ have in the interval $[0, 2\pi]$?

   (a) 0

   (b) 1

   (c) 2

   (d) 3

   (e) 4

Solution: E. $f(x)$ can be rewritten $(\sin^2(x) - \cos^2(x))(\sin^2(x) + \cos^2(x)) = \sin^2(x) - \cos^2(x) = -\cos(2x)$. Since $\cos(x)$ has two roots in the given interval, $\cos(2x)$ has four roots because its period is half that of $\cos(x)$.

8. Consider the following algorithm:

   1. Take an integer x as input.
   2. If x is odd, add one to x. Otherwise, square x and then subtract one from x.
   3. If $x > 2012$, output x and stop. Otherwise, go to line 2.

   For how many input values will the algorithm fail to produce an output?

   (a) 0

   (b) 1

(c) 2

(d) 3

(e) Infinitely many

Solution: C. The "bad" inputs are 0 and -1, which will cause the algorithm to loop, with x oscillating between 0 and -1. To see that the algorithm halts for all other input values, we consider the following cases:

(i) If $x \geq 2$, then $\min(x^2 - 1, x + 1) \geq x$, so the algorithm will cause the value of x to increase until $x > 2012$.

(ii) If $x = 1$, then we add one to get $x = 2$, and we return to the previous case.

(iii) If $x \leq -2$ is even, then $x^2 - 1 \geq 3$, and then we return to case (i).

(iv) If $x \leq -2$ is odd, then $x + 1 \leq -2$, in which case we return to case (i).

9. A box contains a collection of stamps worth 23 cents and stamps worth 25 cents. The total value of the 23-cent stamps equals the total value of the 25-cent stamps, and the total value of all the stamps in the collection is less than 35 dollars. What is the maximum possible number of stamps in the box?

(a) 96

(b) 119

(c) 121

(d) 144

(e) 192

Solution: D. If x is the number of 23-cent stamps and y is the number of 25-cent stamps, then we have $23x = 25y$ and $23x + 25y < 3500$. We conclude that 23 divides y evenly and that 25 divides x evenly. Thus, if $x = 25k$, then $y = 23k$, which means that the total value of the stamps is of the form $23 \cdot 25k + 25 \cdot 23k = 1150k$ cents; moreover, we have $(25 + 23)k = 48k$ stamps. Taking $k = 3$ maximizes the total value of our stamps (at \$34.50), so our maximum number of stamps is $48 \cdot 3 = 144$.

10. Let C be the intersection of the plane $x + y - z = 0$ and the cone $z^2 = 3x^2 + y^2$. What curve is formed by projecting C onto the xy-plane?

(a) Circle

(b) Ellipse

(c) Hyperbola

(d) Parabola

(e) None of the above

Solution: E. Taking $z = x+y$, we have that $(x+y)^2 = 3x^2+y^2$, or $x^2+2xy+y^2 = 3x^2+y^2$, so the curve must satisfy the relation $2x^2 - 2xy = 0$. Projecting onto the xy-plane is the same as ignoring the z-coordinate, so the projection onto the xy-plane is in fact defined by $2x^2-2xy = 0$, or $2x(x-y) = 0$. The curve defined by this equation is a pair of intersecting lines: $x = 0$ and $x = y$.

11. Let $f(x) = \cos^2(x)$, $g(x) = 1/x$, and $h(x) = \sin(x)$. What is $f \circ (g^{-1} \circ h)^{-1}(2)$?

(a) $\frac{1}{3}$

(b) $\frac{3}{4}$

(c) 1

(d) $\frac{4}{3}$

(e) 3

Solution: B. Since $g^{-1}(x) = 1/x$, $(g^{-1} \circ h)^{-1}(x) = (g \circ h)^{-1}(x) = \text{arccsc}(x)$. Then $f \circ \text{arccsc}(x) = 1 - \frac{1}{x^2}$, so evaluating the function at $x = 2$ gives $\frac{3}{4}$.

12. Let x and y be two-digit integers whose mean is 80. What is the minimum possible value of xy?

(a) 5916

(b) 5959

(c) 6000

(d) 6039

(e) None of the above

Solution: D. Since they have mean 80, we have that $x + y = 160$, and since $x, y < 100$, we have $x, y > 60$. To minimize xy, we write $y = 160 - x$, so $xy = 160x - x^2$. This is a downward-opening parabola, so its minimum values occur at the extreme values of x: $160(61) - (61)^2 = 99 \cdot 61$ and $160(99) - (99)^2 = 61 \cdot 99$, both of which are 6039.

13. What is the units digit of $2012^{2012^{2012}}$ in base ten?

(a) 0

(b) 2

(c) 4

(d) 6

(e) 8

Solution: D. We need only look at the units digit of 2012^k to observe how it behaves as k grows. If $k \equiv 1 \bmod 4$, the units digit is 2; if $k \equiv 2$, the units digit is 4; if $k \equiv 3$, the units digit is 8; and if $k \equiv 0$, the units digit is 6. Now observe that if $k = 2012^{2012}$, then $k \equiv 0 \bmod 4$, so the units digit of $2012^{2012^{2012}}$ is 6.

14. A circle has area $\ln(a^3)\ln(b^3)$ and circumference $\ln(b^2)$, where ln is the natural logarithm. What is $\log_b(a)$?

(a) $\frac{1}{9\pi}$

(b) $\frac{3}{2\pi}$

(c) $\frac{2\pi}{3}$

(d) $\frac{9}{\pi}$

(e) 9π

Solution: A. Recall the formulas $A = \pi r^2$ and $C = 2\pi r$, where A and C are respectively the area and circumference of a circle of radius r. The relationship between A and C is therefore given by $A = C^2/4\pi$. Inputting the appropriate values for A and C gives $\ln(a^3)\ln(b^3) = \ln^2(b^2)/4\pi$, which simplifies to $9\ln(a)\ln(b) = \ln^2(b)/\pi$, or $\ln(a)/\ln(b) = 1/9\pi$. By the change-of-base formula, $\log_b(a) = \ln(a)/\ln(b)$, so $\log_b(a) = 1/9\pi$.

15. A certain shipping company requires packages to satisfy the restriction that the *length* plus the *girth* of the package not exceed 120 inches. (The length is the longest dimension of the package, and the girth is the distance around the two smaller sides of the package.) What is the volume of the largest cubical package that the company will ship?

(a) 8000 in.^{3}

(b) 13824 in.^{3}

(c) 15625 in.^{3}

(d) 24389 in.^{3}

(e) 27000 in.^{3}

Solution: B. Let ℓ be the length of a side of the cubical package. Then the length of the package is ℓ and the girth is 4ℓ, so we must have $5\ell \leq 120$. We maximize volume by maximizing ℓ, so take $\ell = 24$; the volume is then $24^3 = 13824$.

16. Charles can wash the car in 90 minutes. Patrick can wash the car in 2 hours. Rachel has never washed the car before but wants to help today. All three of them work together and wash the car in 36 minutes. At the rate Rachel was going, how long would it have taken her to wash the car by herself?

    (a) $\frac{16}{17}$ hours

    (b) $\frac{3}{2}$ hours

    (c) $\frac{17}{10}$ hours

    (d) 2 hours

    (e) None of the above

Solution: D. Reduce everything to the same units. Per hour, Charles can wash $2/3$ of the car, Patrick can wash $1/2$ of the car, and Rachel can wash $1/x$ of the car, where x is how long (in hours) she would take to wash 1 car. If they get the car washed in 36 minutes, that's $3/5$ an hour, so they could wash $5/3$ cars in one hour. Solve $\frac{2}{3} + \frac{1}{2} + \frac{1}{x} = \frac{5}{3}$ for x to get $x = 2$. So Rachel could wash the car by herself in 2 hours.

17. A fair six-sided die (with sides labeled from 1 to 6) and a fair ten-sided die (with sides labeled from 1 to 10) are rolled independently. The six-sided die lands on x, and the ten-sided die lands on y. What is the probability that $|x - y| < 3$?

    (a) $\frac{1}{3}$

    (b) $\frac{7}{20}$

    (c) $\frac{9}{20}$

    (d) $\frac{1}{2}$

    (e) $\frac{31}{60}$

Solution: C. If $x = 1$, there are precisely three values of y less than three away from x: $1, 2, 3$. If $x = 2$, there are four values: $1, 2, 3, 4$. If $x > 2$, then there are precisely five values that y can be. Thus, there are $3 + 4 + 5 + 5 + 5 + 5 = 27$ ways out of 60 for $|x - y| < 3$.

18. What is the maximum number of regions a plane can be divided into by 100 straight lines?

    (a) 4950

(b) 4951

(c) 5050

(d) 5051

(e) None of the above

Solution: D. The formula for the maximum number of regions can be divided into by n lines is $\frac{n(n+1)}{2} + 1$; for $n = 100$, this is equal to 5051.

To derive this formula, proceed inductively: given $n-1$ lines, we want to pick a straight-line path through as many regions as possible. For each region we enter, our path will divide the region into two regions, adding one region to our total number. To enter a new region, we must cross a line; since two lines can meet in at most one point, we enter at most $n-1$ new regions *plus* the region we're in before crossing any lines; thus we enter (and gain) n regions total. If $R_{n-1} = \frac{(n-1)n}{2} + 1$ is the maximum number of regions we can have with $n-1$ lines, then

$$R_n = \left(\frac{(n-1)n}{2} + 1\right) + n = \frac{(n-1)n + 2n}{2} + 1 = \frac{(n-1+2)n}{2} + 1 = \frac{n(n+1)}{2} + 1,$$

as we claimed. Since the formula holds for the base case $n = 0$ (with no lines, we have one region), we are done.

19. How many words of length 10 using only the letters 'G' and 'T' are there that have no consecutive 'G's?

(a) 128

(b) 132

(c) 136

(d) 140

(e) 144

Solution: E. There can be at most 5 'G's in the word. For $k = 0, 1, 2, 3, 4, 5$ choose a subset of $\{1, 2, ..., 10\}$ of size k containing no consecutive integers. The number of such subsets is $\binom{n-k+1}{k}$, where $n = 10$. This gives

$$\binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5} = 1 + 10 + 36 + 56 + 35 + 6 = 144.$$

20. Lockers labeled from 1 to 100 are lined up in a school. All the lockers' doors are open. A student walks by and closes all the lockers. Another student walks by and opens all lockers

whose labels are divisible by two. Another student walks by and switches the state of lockers (that is, the student opens closed lockers and closes opened lockers) with labels divisible by three. This process continues, with the n^{th} student switching the state of lockers with labels divisible by n. After 100 students have walked by, how many lockers are closed?

(a) 0

(b) 10

(c) 12

(d) 13

(e) 100

Solution: B. The n^{th} locker is "toggled" once for every positive divisor of n. Writing the prime factorization of n as $p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, n has $(\alpha_1 + 1) \cdots (\alpha_k + 1)$. If α_i is odd for any i, then the number of divisors is even. Only locker numbers with an odd number of divisors will be closed at the end; for such n, each α_i must be even. Thus, $n = p_1^{2\cdot\beta_1} \cdots p_k^{2\cdot\beta_2} = \left(p_1^{\beta_1} \cdots p_k^{\beta_k}\right)^2$. So locker n is closed if and only if n is a perfect square. There are ten perfect squares from 1 to 100, so ten lockers will be closed.

21. The edges of the hexagon $ABCDEF$ are each to be colored either red, blue, or green. How many distinct colorings are possible if no two adjacent edges may have the same color?

(a) 66

(b) 72

(c) 81

(d) 96

(e) None of the above

Solution: A. An equivalent question is to ask how many ways there are to make six-letter words from the letters $\{a, b, c\}$ so that no two consecutive letters are the same—there are $3 \cdot 2^5 = 96$ such words—and then subtract the number of such words where the first and last letter are the same. For these, first consider words of the form aXa. The X can be replaced by the following strings starting with b: $babc, bacb, bcab, bcac, bcbc$. Swapping the letters b and c gives the five strings starting with c. Since we have analogous situations for words of the form bXb and cXc, there are 30 words we need to subtract, giving us our final answer of $96 - 30 = 66$.

22. Two circles of diameter 1 are placed side-by-side in a 2-by-1 rectangle. What is the side length of the largest equilateral triangle that can be fit into the rectangle without overlapping the circles?

(a) $\frac{\sqrt{6}-\sqrt{3}}{3}$

(b) $1-\frac{\sqrt{3}}{3}$

(c) $\frac{\sqrt{3}}{4}$

(d) $2-\frac{2\sqrt{3}}{3}$

(e) $\frac{\sqrt{3}}{2}$

Solution: B. The largest triangle is tangent to the two circles and has its third side along the edge of the rectangle (take it to be the bottom edge). The left side of the triangle is angled $\pi/3$ above the horizontal, so the radius of the left circle to the tangent point is $\pi/6$ below horizontal. The tangent point is therefore at a height of $1/4$ above the bottom of the rectangle, and $1/2-\sqrt{3}/4$ left of the center of the rectangle. Using the fact that the left side of the triangle has slope $\sqrt{3}$, the left most vertex of the triangle is $\sqrt{3}/12$ left of the tangent point, and so $1/2-\sqrt{3}/6$ left of the center. The total length of the bottom side is then $1-\sqrt{3}/3$.

23. Let $a<b$ be positive integers satisfying $a^b=b^a$. How many pairs of values are possible for a and b?

(a) 0

(b) 1

(c) 2

(d) 3

(e) Infinitely many

Solution: B. Let $f(x)=\frac{\ln(x)}{x}$. The requirement that $a^b=b^a$ is equivalent to $f(a)=f(b)$. The only critical point of $f(x)$ is at $x=e$, which is a maximum. $f(1)=0$ and f is increasing from 1 to e. $f(x)$ decreases asymptotically to 0 for $x>e$. Therefore the only possible integer value of a is 2. If $a=2$, then $b=4$. Therefore there is one solution.

24. What is the value of the infinite sum $\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k k}{1-4k^2}$?

(a) $\frac{3}{16}$

(b) $\frac{1}{5}$

(c) $\frac{2}{9}$

(d) $\frac{3}{10}$

(e) None of the above

Solution: E. Observe that $1-4k^2 = (1-2k)(1+2k)$. Using partial fraction decomposition, we have

$$\frac{k}{1-4k^2} = \frac{1}{4(1-2k)} - \frac{1}{4(1+2k)}.$$

Thus,

$$\sum_{k=0}^{\infty} \frac{(-1)^k k}{1-4k^2} = \frac{1}{4}\sum_{k=0}^{\infty} \frac{(-1)^k}{1-2k} - \frac{1}{4}\sum_{k=0}^{\infty} \frac{(-1)^k}{1+2k}.$$

Making the change of index $j = k+1$ in the second sum gives

$$\frac{1}{4}\sum_{k=0}^{\infty} \frac{(-1)^k}{1-2k} - \frac{1}{4}\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{1+2(j-1)} = \frac{1}{4}\sum_{k=0}^{\infty} \frac{(-1)^k}{1-2k} - \frac{1}{4}\sum_{j=1}^{\infty} \frac{(-1)^j}{1-2j}.$$

Thus, only the $k=0$ term of the first sum survives cancellation, leaving the answer as $\frac{1}{4} \cdot \frac{1}{1} = \frac{1}{4}$.

25. Two ants are on the xy-plane, one positioned at $(-1,1)$ and the other at $(2,0)$. There is a puddle occupying the region $x^2+y^2<1$. What is the length of the shortest path from one ant to the other which does not pass through the puddle?

(a) $1 + \frac{\pi}{6} + \sqrt{3}$
(b) $\sqrt{10}$
(c) $5 - \sqrt{3}$
(d) $\frac{3\pi}{4} + \sqrt{2}$
(e) $1 + 3e - \pi$

Solution: A. The shortest path is along a tangent line from the ant at $(-1,1)$ to the unit circle $C : x^2+y^2 = 1$, along the circumference of C, and then along a tangent line from C to the other ant at $(2,0)$. A tangent line to C which passes through $(-1,1)$ is $y=1$, hitting the point $(0,1)$. The distance traveled along the line $y=1$ is 1.

A tangent line from $(2,0)$ to C is given by choosing m so that

$$y = m(x-2) \quad \text{and} \quad x^2+y^2 = 1$$

have one point of intersection. This occurs when

$$x^2 + m^2(x-2)^2 = 1 \implies (1+m^2)x^2 - 4m^2x + 4m^2 - 1 = 0$$

has a double root, i.e., when the discriminant is zero:

$$b^2 - 4ac = 16m^4 - 4(1+m^2)(4m^2-1) = -12m^2 + 4 = 0,$$

so $m = \pm 1/\sqrt{3}$. We'll take the negative value for m (so that the point we intersect will be closer to $(0,1)$), so the root is

$$x = -b/2a = \frac{4m^2}{2(1+m^2)} = \frac{4/3}{2(1+1/3)} = \frac{1}{2},$$

so $y = -\frac{1}{\sqrt{3}}\left(\frac{1}{2} - 2\right) = \frac{\sqrt{3}}{2}$. The segment traveled along this line $y = \frac{1}{\sqrt{3}}(x-2)$ is a leg of the 30-60-90 triangle with hypotenuse 2 (the segment between $(0,0)$ and $(2,0)$) and other leg 1 (the radius of C), the segment has length $\sqrt{3}$.

Finally, since the point $(1/2, \sqrt{3}/2)$ on the unit circle corresponds to an angle of $\pi/3$ from the x-axis and the point $(0,1)$ corresponds to an angle of $\pi/2$, the points differ by an angle (hence distance along the circumference of C) of $\pi/6$. Thus, the total distance traveled is $1 + \pi/6 + \sqrt{3}$.

26. Onewayville's downtown region consists of a grid of 5 one-way streets running north to south and 5 one-way streets running west to east. How many routes are there from the northwest corner to the southeast corner which avoid the intersection of the middle north-south street with the middle west-east street?

    (a) 34
    (b) 36
    (c) 70
    (d) 256
    (e) None of the above

Solution: A. In an grid with n streets by n streets, the number of paths from the northwest to southeast corner is $\binom{2(n-1)}{n-1}$ (to see this, imagine flipping a coin at each intersection to determine whether to go south or east: we have $2(n-1)$ coin flips, $n-1$ of which have to come up "south" and $n-1$ of which have to come up "east" in order to end at the southeast corner). The total number of paths from the northwest to the southeast is therefore $\binom{8}{4} = 70$. The number of paths that pass through the center point is $\binom{4}{2} \cdot \binom{4}{2} = 36$ (to see this, imagine combining two smaller 3×3 grids: one from the northwest corner to the center, and then one from the center to the southeast corner). The total number of paths avoiding the center point is therefore $70 - 36 = 34$.

27. Brittany and Chris are throwing a party. In total, they know 7 other men and 7 other women. How many different sets of friends can they invite so that they invite the same number of men and women to the party? (Assume that they invite at least one person to the party.)

    (a) 127

(b) 254
(c) 3431
(d) 8191
(e) None of the above

Solution: C. There are $\binom{7}{k}^2$ ways of inviting k men out of 7 and k women out of 7. Summing k from 1 to 7, we have

$$\binom{7}{1}^2 + \binom{7}{2}^2 + \binom{7}{3}^2 + \binom{7}{4}^2 + \binom{7}{5}^2 + \binom{7}{6}^2 + \binom{7}{7}^2 = \binom{14}{7} - 1 = 3431.$$

(using the identity $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$).

28. An *Albert square* is formed from an $n \times n$ grid of zeros by repeatedly adding or subtracting 1 from every cell in a row or column. For example, $\begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$ is an Albert square because it can be formed from a 2×2 grid of zeros by adding 2 to each cell in the first row, subtracting 2 from each cell in the first column, and adding 1 to each cell in the second row.
If $\begin{bmatrix} 20 & 12 & a \\ 2 & b & 0 \\ c & d & 12 \end{bmatrix}$ is an Albert square, what is $a + b + c + d$?
(a) -12
(b) 0
(c) 20
(d) 32
(e) None of the above

Solution: D. First, note that the sum of any three entries that share no column or row must always be the same. To see this, let r_1, r_2, r_3 denote the number of times 1 is added to (or subtracted from) rows 1, 2, and 3, and similarly, let c_1, c_2, c_3 denote the number of times a 1 is added to (or subtracted from) a column. Then, if the (i, j)th entry in our matrix is denoted $a_{i,j}$, we have that $a_{i,j} = r_i + c_j$. Thus, if three entries, say u, s, t, share no column or row, then $u + s + t = r_1 + r_2 + r_3 + c_1 + c_2 + c_3$. Now, $a_{1,2} + a_{2,1} + a_{3,3} = 26$. Filling in the rest of the entries implies $a + b + c + d = 18 + -6 + 14 + 6 = 32$.

29. A square with side length 1 is cut into nine equal squares to form a 3×3 grid. The four corner squares are shaded, and the middle square is divided into a 3×3 grid as before. The four corner squares of the new grid are shaded and the middle square is subdivided in the same way. The pattern continues inward infinitely. What is the area of the shaded region?

(a) $\frac{1}{2}$

(b) $\frac{\sqrt{2}}{3}$

(c) $\frac{29}{20}$

(d) $\frac{17}{36}$

(e) None of the above

Solution: A. Each of the four largest shaded squares have area $\frac{1}{9}$ of the square's total area. The next four largest shaded squares have area $\frac{1}{9}$ of the total area of the middle square This pattern continues, where the i^{th}-largest squares will have one-ninth of the area of the $(i-1)^{\text{th}}$-largest squares, and there are four of them. Thus the total shaded area is

$$4\left(\frac{1}{9}\right) + 4\left(\frac{1}{9}\cdot\frac{1}{9}\right) + 4\left(\frac{1}{9}\cdot\frac{1}{9}\cdot\frac{1}{9}\right) + \cdots = \frac{4}{9}\sum_{k=0}^{\infty}\left(\frac{1}{9}\right)^k.$$

This is a geometric series, so we get

$$\frac{4}{9}\cdot\frac{1}{1-1/9} = \frac{1}{2}.$$

30. Let $N > 1$ be an integer. Suppose you spin a spinner with landing spots marked by the positive integers which evenly divide N, each equally likely to be landed upon. Let d be the number the spinner lands on.

- If d is the product of an even number of distinct primes, then you win $\frac{N}{d}$ dollars. (For our purposes, 1 is the product of zero distinct primes.)
- If d is the product of an odd number of distinct primes, then you lose $\frac{N}{d}$ dollars.
- If d is a multiple of a perfect square greater than 1, then you neither gain nor lose money.

For which of the following choices of N is your expected winnings greatest?

(a) 50

(b) 66

(c) 80

(d) 96

(e) 110

Solution: E. Given N, the expected winnings for one spin on the spinner is

$$\frac{1}{\tau(N)}\sum_{d|N}\mu(d)\frac{N}{d},$$

where

$$\mu(d)=\begin{cases}(-1)^k, & d \text{ is the product of } k \text{ distinct primes}\\ 0, & d \text{ is divisible by a square} > 1,\end{cases}$$

where $\tau(N)$ is the number of divisors of N (i.e. positive integers dividing N evenly), and where $\sum_{d|N}$ means the sum over the divisors of N.

Now we use two standard facts from number theory. First, if $\varphi(n)$ is the Euler totient function (which counts the number of integers from 1 to n which are relatively prime to n), then

$$\sum_{d|n}\varphi(d)=n;$$

second, the Möbius inversion formula:

$$\text{if } g(n)=\sum_{d|n}f(d), \text{ then } f(n)=\sum_{d|n}\mu(d)g(n/d).$$

Thus, taking $f(n)=\varphi(n)$ and $g(n)=n$, the formula for expected winnings reduces to

$$\frac{\varphi(N)}{\tau(N)}.$$

We can now compute $\varphi(N)/\tau(N)$ for each choice using the facts that φ and τ are multiplicative (f is multiplicative if $f(ab)=f(a)f(b)$ whenever a and b are relatively prime) and that $\varphi(p^k)=p^k-p^{k-1}$ and $\tau(p^k)=k+1$ for all primes p.

- $\varphi(50)/\tau(50)=\varphi(2)\varphi(5^2)/\tau(2)\tau(5^2)=\frac{20}{6}=3.333...$
- $\varphi(66)/\tau(66)=\varphi(2)\varphi(3)\varphi(11)/\tau(2)\tau(3)\tau(11)=\frac{20}{8}=2.5$
- $\varphi(80)/\tau(80)=\varphi(2^4)\varphi(5)/\tau(2^4)\tau(5)=\frac{32}{10}=3.2$
- $\varphi(96)/\tau(96)=\varphi(2^5)\varphi(3)/\tau(2^5)\tau(3)=\frac{32}{12}=2.666...$
- $\varphi(110)/\tau(110)=\varphi(2)\varphi(5)\varphi(11)/\tau(2)\tau(5)\tau(11)=\frac{40}{8}=5$

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