Problem #1

For how many integers is $x^4 + x^3 + x^2 + x + 1$ a perfect square?

Solution: The answer is 3. The solution $x = 0$ is a solution because it gives a perfect square of 1. If x is even notice that $(x^2+x/2)^2 < x^4+x^3+x^2+x+1 < (x^2 + x/2 + 1)^2$. When x is odd, it must be that $x^4+x^3+x^2+x+1 = (x^2+x/2+1/2)^2$. This reduces to $x^2 - 2x - 3 = 0$. So the only other solutions are $x = -1$ and $x = 3$. This gives a total of three solutions.

Answer:

Problem #2

The perimeter of parallelogram $ABCD$ is 30 and its altitudes are of length 4 and 7. What is $\sin A$?

Solution: There are two ways we can obtain the area of $ABCD$. Let x be one side's length. Let $15-x$ be the other side length. The area is equal to both $7x$ and $4(15-x)$. Equating these two expressions gives $x = 60/11$. Then $\sin A = 4/x = 4/(60/11) = 11/15$. Notice that the sine of each angle is the same since the sine function is symmetric about 90 degrees.

Answer:

Problem #3

The number $(9^6 + 1)$ is the product of three primes. What is the largest of these three primes?

Solution: Notice that $(9^6 + 1) = (9^2 + 1)(9^4 - 9^2 + 1) = 82 \cdot 6481 = 2 \cdot 41 \cdot 6481$. So the answer is 6481.

Answer:

Problem #4

Simplify the following:

$$\sum_{i=2}^{2010} \frac{1}{k^2 - k}$$

Solution: The solution is 2009/2010. Use partial fractions to obtain a telescoping sum.

Answer:

Problem #5

What is $\lfloor \sqrt{n^2 - 18n + 80} \rfloor$ when $n = 12345678$? Here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Solution: Here $\lfloor \sqrt{n^2 - 18n + 80} \rfloor = \lfloor \sqrt{(n-9)^2 - 1} \rfloor$. So the answer is 12345668.

Answer:

Problem #6

What is the units digit of $9^{12}13^{13}17^{14}$?

Solution: Notice that 9^{12} ends in a 1 and $13 \cdot 17$ ends in a 1. Thus some integer ending in 1 is multiplied by 17. This gives a units digit of 7.

Answer:

Problem #7

Find a positive integer not exceeding 1000 which leaves a remainder of 3 upon division by 7, 4 upon division by 11 and 2 upon division by 13.

Solution: The answer is 444.

Answer:

Problem #8

Simplify $\sqrt{4+\sqrt{7}} - \sqrt{4-\sqrt{7}}$:

<u>Solution:</u> Let $x = \sqrt{4+\sqrt{7}} - \sqrt{4-\sqrt{7}}$. Then $x^2 = 4+\sqrt{7} - 2\sqrt{4+\sqrt{7}}\sqrt{4-\sqrt{7}} + 4 - \sqrt{7} = 8 - 2\sqrt{9} = 2$. So the answer is $\sqrt{2}$.

Answer:

Problem #9

Triangle ABC has $\tan(A) = 3/4$ and $\tan(B) = 21/20$. What is AC/BC?

Solution: By the Law of Sines, we know that $\sin(A) = 3/5$ and $\sin(B) = 21/29$. So $AC/BC = (21/29)(5/3) = 35/29$. So the answer is $35/29$.

Answer:

Problem #10

The square below can be filled in so that each row and each column contains each of the numbers 1, 2, 3 and 4 exactly once. What does x equal?

			1
	2		
		x	
1			4

Solution: Notice that the bottom row is forced, then the second column is forced, then the top row, and finally the third column. This gives $x = 4$.

Answer:

2	4	3	1
	2	1	
	1	x	
1	3	2	4