Georgia Institute of Technology

High School Mathematics Competition 2010

Junior Varsity Proof-Based Test
Problem #1

ID#:

Score:

Prove that for every prime p except 2 and 5 there exists some multiple of p that ends with the digits 0001.

Solution: Consider the sequence $1, p, p^2, p^3, \ldots$. Each term p^k can only have one of 10000 different remainders upon division by 10000, but the sequence contains an infinite number of terms, so there exist i and j such that $1 \leq i < j$ and $p^i \equiv p^j \mod 10^4$. In other words, $p^j - p^i = p^i(p^{j-i} - 1)$ is a multiple of 10000. But if p is any prime other than 2 or 5, then p^i is relatively prime to 10000. Therefore $p^{j-1} - 1$ is a multiple of 10000, which means that p^{j-i} ends in the digits 0001.

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Problem #2

ID#:

Score:

Prove that if P is any point in an equilateral triangle of height h then the sum of the perpendicular distances from P to the three sides of the triangle is equal to h.

Solution: Join point P to the vertices of the triangle. This creates three triangular regions. Consider the area of each triangle. The formula for the area of a triangle is $\frac{1}{2}bh$. Notice that the base of these three triangles is the same as the base of the entire triangle, which also has an area of $\frac{1}{2}bh$. Since the area of the sum of the smaller triangles is the same as the entire triangle, the sum of the heights (i.e. perpendicular distances) must be equal to h. This completes the proof.

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Junior Varsity Proof-Based Test
Problem #3

ID#:
Score:

The points of a plane are colored three colors. Show there exist two points with distance one that both have the same color.

Solution: Let the colors be black, white and red. Suppose that any two points with distance one have different colors. Choose any red point r and assign to it the figure below. One of the two points b and w must be white and the other black. Hence, the point r' must be red. Notice that if we rotate the figure about r we get a circle of red points r'. This circle contains a chord of length 1, a contradiction.

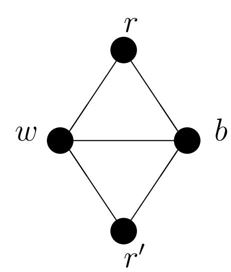

Figure 1: The figure.

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Junior Varsity Proof-Based Test
Problem #4

ID#:

Score:

Show that if n is a positive integer then

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

Solution: We will prove this using induction. We first prove the case $n = 1$ (base case). Here $1 < 2\sqrt{1} = 2$. Now suppose that the inequality holds for some integer k. So,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} < 2\sqrt{k}$$

We will show that it holds for $n = k + 1$. So from our assumption, we have

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k} + \frac{1}{\sqrt{k+1}}$$

As a result, it suffices to show that $2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$. If we multiply both sides by $\sqrt{k+1}$, we get $2\sqrt{k(k+1)} + 1 < 2(k+1) = 2k + 2$. This simplifies to $2\sqrt{k(k+1)} < 2k + 1$. Squaring both sides gives $4(k(k+1)) < 4k^2 + 4k + 1$. This inequality is true. As all our steps were reversible, we have that the inequality $2\sqrt{k} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$ is true and this proves our induction.

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Junior Varsity Proof-Based Test
Problem #5

ID#:
Score:

Show that the points w, x, y, z are the vertices of a parallelogram in the complex plane *if and only if* the sum of some two of them is equal to the sum of the other two. (Assume that not all four points are collinear.)

Solution: Let w, x, y and z be vertices of a parallelogram in the complex plane, where w and y are opposite vertices, so x and z are also opposite vertices. We know that in every parallelogram, the diagonals bisect each other. So the midpoints of each diagonal coincide. Therefore $\frac{w+y}{2} = \frac{x+z}{2}$ and so $w + y = x + z$. So we have shown that if w, x, y, z are the vertices of a parallelogram in the complex plane, then the sum of some two of them equals the sum of the other two.

We now must show the converse of this statement, namely that if there are four points such that the sum of two of them is equal to the sum of the other two, then the four points form a parallelogram. Suppose, without loss of generality that $w + y = x + z$. Then $(w + y)/2 = (x + z)/2$. This means the midpoint of the line segment between w and y and the midpoint of the line segment between x and z coincide. In other words, in the quadrilateral formed by the complex numbers w, x, y and z, the diagonals bisect each other. Thus the quadrilateral formed by these four points is a parallelogram.