Georgia Tech HSMC 2009

Varsity Multiple Choice

February 28^{th}, 2009

Note: The cut-off to advance to the Proof exam was 13/20.

1. How many perfect cubes greater than 1 divide 9^9?

   (a) 4
   (b) 6
   (c) 7
   (d) 8
   (e) 10

   Solution: (B) There are six perfect cubes, and they are $3^3, 3^6, 3^9, 3^{12}, 3^{15}, 3^{18}$.

2. A cube and sphere have the same surface area. What is the ratio of the volume of the sphere to that of the cube?

   (a) $\frac{\pi}{2}$
   (b) $\sqrt{\frac{3}{\pi}}$
   (c) $\sqrt{\frac{6}{\pi}}$
   (d) $\frac{4}{3}\pi$
   (e) $\pi\sqrt{2}$

   Solution: (C) Let x be the side length of the cube and r be the radius of the sphere. Then $6x^2 = 4\pi r^2$. So $r = x\sqrt{3/2\pi}$. Plugging this value for r into the formula for the volume of a sphere, $\frac{4}{3}\pi r^3$ gives the answer of $\sqrt{\frac{6}{\pi}}$.

3. What is the remainder when $1 + x^3 + x^6 + x^9 + x^{27}$ is divided by $x^2 - 1$?

   (a) 1

(b) 3

(c) $x + 2$

(d) $3x + 2$

(e) $4x + 1$

Solution: (D) We have $1 + x^3 + x^6 + x^9 + x^{27} = q(x)(x^2 - 1) + ax + b$. Set $x = 1, x = -1$ and this gives a system of simultaneous equations, $5 = a + b$ and $-1 = -a + b$. Then $a = 3, b = 2$, which gives $3x + 2$ as the remainder.

4. How many zeroes are at the end of 35! ?

(a) 6

(b) 7

(c) 8

(d) 9

(e) 10

Solution: (C) Notice that the only way we can get a zero in this expression is for there to be a factor of $10 = 2 \cdot 5$. The limiting constraint is the number of 5's and there are 8 factors of 5 in 35!, so the answer is 8.

5. Three fair 10-sided dice, each labeled $1, 2, \ldots, 10$ are tossed. One is colored red, one is colored blue and one is colored yellow. What is the probability that the numbers at the top of each die satisfy the inequality: red > blue > yellow ?

(a) $\frac{1}{6}$

(b) $\frac{1}{8}$

(c) $\frac{3}{25}$

(d) $\frac{2}{15}$

(e) $\frac{6}{25}$

Solution: (C) The probability that the numbers on the three dice are all distinct is $1 \cdot \frac{9}{10} \cdot \frac{8}{10} = \frac{18}{25}$, and the probability that the ordering is correct is $\frac{1}{6}$, so the final answer is $\frac{18}{25} \cdot \frac{1}{6} = \frac{3}{25}$.

6. Suppose that y is a real number that satisfies

$$3^{3^y} + 9^{3^y} = 30.$$

What is the value of $3^{3^{3^y}}$?

(a) 50

(b) 120

(c) 144

(d) 243

(e) 729

Solution: (D) Let $x = 3^{3^y}$. Then our equation is $x + x^2 = 30$. So $3^{3^y} = 5$. Then we have that $3^{3^{3^y}} = 3^5 = 243$.

7. How many ordered pairs of integers (x, y) are solutions to

$$x^3 + y^4 = 2009?$$

(a) 0

(b) 1

(c) 2

(d) 3

(e) 4

Solution: (A) Notice that 2009 is congruent to 7 modulo 13. The residues of a cube mod 13 are 0,1,5, 8 or 12, while the residues of a fourth power are 0,1,3 or 9. Notice then there are no integral solutions to our equation.

8. How many integer solutions (x, y) are there for $(x-2)(x-10) = 3^y$?

(a) 1

(b) 2

(c) 3

(d) 4

(e) 5

Solution: (B) Notice that we need two powers of 3 that differ by 8. Such sets of powers are (1,9) and (-1, -9). So the only two solutions for (x, y) and $(11, 2)$ and (1,2).

9. In Kazakhstan, Borat can buy stamps with values of 6 krona, 9 krona and 20 krona. What is the largest postage amount that can not be exactly contained by using stamps with these denominations?

(a) 37

(*b*) 41

(*c*) 43

(*d*) 47

(*e*) 51

Solution: (C) The answer is 43. This is a special case of the postage stamp problem. Notice that $36 = 9{\cdot}4, 38 = 20{+}9{+}9, 40 = 20{+}9{+}9{+}2$, so all the even numbers 36 and up are covered. This also means, by adding a 9, that all numbers over 45 are covered. Notice that 43 can not be obtained from the set of 6, 9 , 20. Thus the answer is C.

10. The average of three positive integers is 5. The average of their reciprocals is 17/72. Their product is 96. What is the median of the three numbers?

(*a*) 3

(*b*) 4

(*c*) 5

(*d*) 6

(*e*) 7

Solution: (B) Guess and check works well here. What three integers have a product of 96 whose average is 5? Notice $96 = 3 \cdot 2^5$. The three numbers are 3, 4 and 8. This gives a median of 4, product of 96, and average of reciprocals $1/8 + 1/4 + 1/3 = 17/(24 \cdot 3)$ is 17/72. More explicitly, consider the factors of 96. In order for the average of the three integers to be five, the largest factor we can use is 12, but this forces us to use (12, 2, 1) as the three integers, and their product is 24. So this leaves 1,2,3,4, 6, 8 as the choices for the integers. The only triple left that gives a product of 96 is (3, 4, 8).

11. Consider 6 distinct points in the plane. If we draw line segments (possibly curved) between each pair of points, what is the minimum number of times these segments cross?

(*a*) 0

(*b*) 1

(*c*) 2

(*d*) 3

(*e*) 4

Solution: (D) This question asks for the *crossing number* of K_6, the complete graph on 6 vertices. The answer is 3. Reference: Relations Between Crossing Numbers of Complete and Complete Bipartite Graphs R. Bruce Richter and Carsten Thomassen, *The American Mathematical Monthly*, Vol. 104, No. 2 (Feb., 1997), pp. 131-137

12. Simplify the following:

$$\sqrt{\frac{16^{10}+8^{12}}{16^5+8^8}}$$

   (a) $16\sqrt{2}$
   (b) 70
   (c) 128
   (d) 200
   (e) 256

Solution: (E) Simplifying this expression gives $\sqrt{\frac{2^{36}(1+2^4)}{2^{20}(1+2^4)}} = \sqrt{2^{16}} = 2^8 = 256$.

13. How many distinct triangles with positive integer sides have perimeter equal to 80?

   (a) 100
   (b) 133
   (c) 147
   (d) 171
   (e) 190

Solution: (B) An enumeration yields the solution of 133. First notice that the maximum side-length is 39. There are 19 different sets of lengths for valid triangles with maximum length 39. There are 18 for maximum side-length 38. There are 16 for maximum side-length 37, and so on. This yields a sum of $19 + 18 + 16 + 15 + 13 + 12 + \cdots + 1 = 133$.

14. How many ordered triples (x, y, z) satisfy the equation:

$$3x^2 + 6y^2 + 4z^2 - 4xy + 4yz = 0?$$

   (a) 0
   (b) 1

(c) 2

(d) 3

(e) infinitely many

Solution: (B) We can factor $3x^2 + 6y^2 + 4z^2 - 4xy + 4yz$ as $(x - 2y)^2 + (y + 2z)^2 + 2x^2 + y^2$. Notice that each of these terms is non-negative, and the only way for all of them to be zero is for x, y and z to be zero. Hence there is only one solution $(0, 0, 0)$.

15. Two trains, 50 miles apart, are traveling towards each other at speeds of 80 and 120 miles per hour, respectively. Meanwhile, Superman is flying between the trains at a speed of 500 miles per hour, and he instantly changes direction when he touches a train. How far does Superman travel before the trains meet?

    (a) 50 miles

    (b) 70 miles

    (c) 100 miles

    (d) 125 miles

    (e) 175 miles

    Solution: (D) The time it takes for the trains to meet is 15 minutes. Superman is traveling at 500 miles per hour, so the total distance is 125 miles.

16. Suppose that $A = 1^{-6} + 2^{-6} + 3^{-6} + \cdots$ and let $B = 1^{-6} + 3^{-6} + 5^{-6} + \cdots$. What is $\frac{A}{B}$?

    (a) $\frac{63}{64}$

    (b) $\frac{127}{128}$

    (c) 1

    (d) $\frac{128}{127}$

    (e) $\frac{64}{63}$

    Solution: (E) Notice that $B = A - (2^{-6} + 4^{-6} + 6^{-6} + \cdots)$. Also $64((2^{-6} + 4^{-6} + 6^{-6} + \cdots) = A$. So the ratio is $64/63$.

17. Twelve people are equally spaced around a large circle. What is the largest number of wires that can be stretched between pairs of people so that no two wires intersect at any point inside the circle? (Also wires can not be routed outside of the circle at any point. However, they can be placed on the boundary of the circle.)

(a) 18
(b) 21
(c) 22
(d) 24
(e) 26

Solution: (B) Inside the circle, there can be at most 9 wires, and there can be 12 wires on the boundary of the circle for a total of 21 wires.

18. One plane flies at a ground speed 75 miles per hour faster than another. On a particular flight, the faster plane requires 3 hours and the slower one 3 hours and 36 minutes. What is the distance of the flight?

(a) 375 miles
(b) 450 miles
(c) 1000 miles
(d) 1350 miles
(e) 1450 miles

Solution: (D) Let d be the distance of the flight and let x be the speed of the slower plane. Solving the equations $\frac{d}{x+75} = 180$ and $\frac{d}{x} = 216$ yields $x = 375$. Solving for d gives 1350 miles.

19. Consider the equation, $x^2 + px + q = 0$. If the roots of this equation differ by 2, then p equals:

(a) $\sqrt{4q+1}$
(b) $2\sqrt{q+1}$
(c) $q+4$
(d) $\sqrt{4q-1}$
(e) $2\sqrt{2q}$

Solution: (B) Suppose the roots are r and $r+2$. Then the sum of the roots is $2r+2 = -p$ and the product of the roots is $r(r+2) = q$. Solving these two simultaneous equations gives $p = 2\sqrt{q+1}$.

20. What is the smallest positive integer $n > 200$ such that $\binom{n}{201}$ is divisible by $\binom{n}{200}$ and these two quantities are not equal?

(a) 202
(b) 230

(c) 402

(d) 500

(e) 602

Solution: (E) Notice that $\binom{n}{201} \,/\, \binom{n}{200} = (n-200)/201$. For the $n = 401$ case, the two quantities are equal. So we need to solve for n such that $(n - 200)/201n = 2$. As a result $n = 602$.