Georgia Institute of Technology

High School Mathematics Competition 2009

Varsity Proof-Based Test
Problem #1

ID#:

In triangle ABC, point E is on $\bar{AB}$, so that $AE = \frac{1}{2}EB$. Find CE if $AC = 4$, $CB = 5$, $AB = 6$.

First Solution: First, we will show a quick solution using Stewart's theorem, we have

$$(AC)^2(EB) + (CB)^2(AE) = (AB)[(CE)^2 + (AE)(EB)]$$

This gives $114 = 6(CE)^2 + 48$, so that $CE = \sqrt{11}$.

Second Solution: A more standard solution uses Heron's formula. Triangles ACE and ACB share the same altitude and $AE = \frac{1}{3}AB$, the area of triangle ACE is equal to one third the area of triangle ACB. By Heron's formula,

$$\frac{1}{3} \text{ the area of } ACB \;= \frac{1}{3}\sqrt{\frac{15}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} = \frac{5}{4}\sqrt{7}.$$

Suppose that $CE = x$. Then the area of triangle ACE is equal to

$$\sqrt{\frac{6+x}{2} \cdot \frac{6-x}{2} \cdot \frac{x+2}{2} \cdot \frac{x-2}{2}}$$

$$= \frac{1}{4}\sqrt{-(x^2-36)(x^2-4)}.$$

Setting $y = x^2$ gives us

$$\frac{5}{4}\sqrt{7} = \frac{1}{4}\sqrt{-(y^2-40y+144)}$$

.

Solving this yields $y = 11$ or 29, and after rejecting 29, we see that $CE = \sqrt{11}$.

Third Solution: By the law of cosines, $(AC)^2+(AE)^2-2(AC)(AE)\cos\alpha = (CE)^2$. Also, $(AC)^2+(AB)^2-2(AC)(AB)\cos\alpha = (CB)^2$. Here $\cos\alpha = 27/48$, and solving for $(CE)^2$ we have that $(CE)^2 = 11$. Thus $\bar{CE} = \sqrt{11}$.

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Problem #2

ID#:

Three sequences, x_n, y_n and z_n, with positive initial terms x_1, y_1, z_1 are defined for $n \geq 1$ by (1) $x_{n+1} = y_n + 1/z_n$, (2) $y_{n+1} = z_n + 1/x_n$, and (3) $z_{n+1} = x_n + 1/y_n$. Show that

(a) None of the three sequences are bounded.

(b) At least one of $x_{200}, y_{200}, z_{200}$ is greater than 20.

Solution: Suppose for purposes of contradiction that one sequence is bounded. Without loss of generality, say this is y_n. So we may assume that x_n and z_n are not bounded. We get a contradiction as y_n is not bounded because of the second equation. Thus, none of the three sequences are bounded.

For the second part, consider the behavior of $a_n^2 = (x_n + y_n + z_n)^2$. Since $x+1/x \geq 2$ for $x > 0$, we observe that $a_2^2 = (x_1+1/x_1+y_1+1/y_1+z_1+1/z_1)^2 \geq 36 = 2 \cdot 18$. Now

$$\begin{aligned} a_{n+1}^2 &= (x_n + y_n + z_n + 1/x_n + 1/y_n + 1/z_n)^2 \\ &> a_n^2 + 2(x_n + y_n + z_n)(1/x_n + 1/y_n + 1/z_n) \\ &\geq a_n^2 + 18 \end{aligned}$$

By induction we get $a_n^2 > 18n$ for $n > 2$. Thus $a_{200}^2 > 3600, x_{200} + y_{200} + z_{200} > 60$. So at least one of $x_{200}, y_{200}, z_{200}$ is greater than 20.

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Varsity Proof-Based Test
Problem #3

ID#:

How many subsets of $S = \{1, 2, \ldots, 17\}$ are there so that each subset contains no two consecutive integers?

Solution: The solution is the 19th fibonacci number, $f_{19} = 4181$. The idea is as follows. We can represent the ith Fibonacci number, f_i where $f_1 = f_2 = 1$, as the number of ways to tile a 1 by $n - 1$ rectangle using 1 by 1 squares and 1 by 2 dominoes. This is easy to see by considering the recurrence relation $f_n = f_{n-1} + f_{n-2}$, where f_{n-1} counts the number of configurations where the last tile was a square and f_{n-2} counts the number of configurations where the last tile was a domino.

For our problem, the idea is that each number in the set S corresponds to the location of a domino on a 1 by 18 rectangle. This gives a one-to-one correspondence between the set $\mathcal{S}$ of all allowable subsets of S and the set of all valid tilings of a 1 by 18 rectangle.

Reference: A. T. Benjamin, J. J. Quinn, *Proofs That Really Count.*

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Varsity Proof-Based Test
Problem #4

ID#:

Suppose that m is a positive odd integer strictly greater than 3. Prove that $\frac{2^{2m}+1}{5}$ is a composite integer.

Solution: Set $2k = m + 1$. Then

$$2^{2m} + 1 = (2^m + 1)^2 - 2^{2k} = (2^m + 2^k + 1)(2^m - 2^k + 1).$$

We now will condition on the value of k mod 4. Notice that each congruence class of k gives a unique congruence class of m mod 4, so we will only refer to the congruence class of k. First, if k is congruent to 2 or 3 mod 4, then $2^m - 2^k + 1 \equiv 0 \bmod 5$. when k is congruent to 0 or 1 mod 4, then $2^m + 2^k + 1 \equiv 0 \bmod 5$.

We also claim that each of the two factors are strictly greater than five. First, observe that it suffices to show that the second factor, $2^m - 2^k + 1$ is strictly greater than five, as the first factor is always larger. When $m > 3$, the first value for m to check is $m = 5$. In this cases, $2^5 - 2^3 + 1 = 25$. It is clear that as m increases, $2^m - 2^k + 1$ also increases, so this completes the proof of the claim.

To see that we obtain a composite integer, observe from the first paragraph that at least one of the two factors is divisible by five and by the second paragraph, both factors are strictly greater than five. Therefore, $\frac{2^{2m}+1}{5}$ is composite

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Varsity Proof-Based Test
Problem #5

ID#:

Prove the following:

$$\frac{1}{\sqrt{1}+\sqrt{4}}+\frac{1}{\sqrt{10}+\sqrt{13}}+\frac{1}{\sqrt{19}+\sqrt{22}}+\frac{1}{\sqrt{28}+\sqrt{31}}+\cdots+\frac{1}{\sqrt{10000}+\sqrt{10003}} > 10$$

.

Solution: This solution is based on creating a telescoping sum and estimating it.

First, observe that $3(\frac{1}{\sqrt{1}+\sqrt{4}} + \frac{1}{\sqrt{10}+\sqrt{13}} + \cdots + \frac{1}{\sqrt{10000}+\sqrt{10003}}) \geq \frac{1}{\sqrt{1}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{7}} + \cdots + \frac{1}{\sqrt{10000}+\sqrt{10003}}$. Rationalizing the denominators of this second sum gives:

$$\frac{1}{\sqrt{1}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{7}}+\cdots+\frac{1}{\sqrt{10000}+\sqrt{10003}} = \frac{\sqrt{4}-\sqrt{1}}{3}+\frac{\sqrt{7}-\sqrt{4}}{3}+\cdots\frac{\sqrt{10003}-\sqrt{10000}}{3}.$$

The theorem holds if the right hand side is at least 30. We have a telescoping sum with value $\frac{\sqrt{10003}-1}{3} > 30$. This proves the result.