

Georgia Tech HSMC 2009

Junior Varsity Multiple Choice

February 28th, 2009

Note: The cut-off for qualification for the Proof test was 11/20.

1. Simplify

$$\frac{\sqrt{\sqrt{10} - 3} + \sqrt{\sqrt{10} + 3}}{\sqrt{\sqrt{10} + 1}}$$

- (a) 1
- (b) $\sqrt{2}$
- (c) $\sqrt{3}$
- (d) 2
- (e) $2\sqrt{2}$

Solution: (B). Squaring the expression and simplifying gives an answer of 2. Notice we take the positive square root because the original expression is clearly positive. Taking the square root then gives $\sqrt{2}$.

2. Suppose we flip two fair coins simultaneously and we continue to do this until at least one of the coins is a head. What is the probability that both coins are heads on this last flip?

- (a) $\frac{1}{6}$
- (b) $\frac{1}{4}$
- (c) $\frac{1}{3}$
- (d) $\frac{1}{2}$
- (e) $\frac{2}{3}$

Solution: (C) The answer is 1/3 because the only three, equally likely outcomes are HH, HT, and TH. The number of TT flips that come before the final set of flips is irrelevant.

3. A cube and sphere have the same surface area. What is the ratio of the volume of the sphere to the volume of the cube?

(a) $\frac{\pi}{2}$

(b) $\sqrt{\frac{3}{\pi}}$

(c) $\sqrt{\frac{6}{\pi}}$

(d) $\frac{4}{3}\pi$

(e) $\pi\sqrt{2}$

Solution: (C) Let x be the side length of the cube and r be the radius of the sphere. Then $6x^2 = 4\pi r^2$. So $r = x\sqrt{3/2\pi}$. Plugging this value for r into the formula for the volume of a sphere, $\frac{4}{3}\pi r^3$ gives the answer of $\sqrt{\frac{6}{\pi}}$.

4. A fair coin is flipped. If it lands heads up, two dice are thrown. If it lands tails up, three dice are thrown. What is the probability that the sum of the numbers showing on the top faces of the dice is 4?

(a) $\frac{7}{144}$

(b) $\frac{1}{18}$

(c) $\frac{27}{432}$

(d) $\frac{1}{16}$

(e) $\frac{5}{72}$

Solution: (A) The probability is given by

$$\frac{1}{2} \cdot \frac{3}{36} + \frac{1}{2} \cdot \frac{3}{216} = \frac{21}{432} = \frac{7}{144}.$$

5. How many zeroes are at the end of $35!$?

(a) 6

(b) 7

(c) 8

(d) 9

(e) 10

Solution: (C) Notice that the only way we can get a zero in this expression is for there to be a factor of $10 = 2 \cdot 5$. We then need to find the minimum of the number of 5's and the number of 2's, and this will be our answer. The number of 5's, is smaller and there are 8 factors of 5 in $35!$, so the answer is 8.

6. One plane flies at a ground speed 75 miles per hour faster than another. On a particular flight, the faster plane requires 3 hours and the slower one 3 hours and 36 minutes. What is the distance of the flight?

- (a) 375 miles
- (b) 450 miles
- (c) 1000 miles
- (d) 1350 miles
- (e) 1450 miles

Solution: (D) Let d be the distance of the flight and let x be the speed of the slower plane. Solving the equations $\frac{d}{x+75} = 180$ and $\frac{d}{x} = 216$ yields $x = 375$. Solving for d gives 1350 miles.

7. What is the smallest positive odd integer n such that the product

$$2^{1/7} 2^{3/7} 2^{5/7} \dots 2^{(2n+1)/7}$$

is greater than 16000?

- (a) 9
- (b) 11
- (c) 13
- (d) 15
- (e) 17

Solution: (A) Recall that $1 = 3 + 5 + \dots + 2n + 1 = (n + 1)^2$. As a result, notice the product $2^{[1+3+\dots+2n+1]/7} = 2^{(n+1)^2/7}$. For $n = 9$, notice that $2^{100/7} > 2^{14} > 16000$.

8. What is the smallest positive integer n such that $\sqrt{n} - \sqrt{n-1} < \frac{1}{60}$?

- (a) 899
- (b) 900
- (c) 901

(d) 2499

(e) 2500

Solution: (C) Notice that we need to find the smallest integer n such that $\sqrt{n} + \sqrt{n-1} > 60$. Since $\sqrt{900} + \sqrt{899} < 60$ and $\sqrt{901} + \sqrt{900} > 60$, $n = 901$.

9. Opposite sides of a regular hexagon are 18 inches apart. The length of each side, in inches is:

(a) 10

(b) $8\sqrt{2}$

(c) $9\sqrt{2}$

(d) $6\sqrt{3}$

(e) $\frac{13}{2}\sqrt{3}$

Solution: (D) Let the vertices of the hexagon be labeled consecutively A_1, A_2, \dots, A_6 . Consider the line segment A_1A_3 , which is of length 18. Draw a perpendicular bisector of this line segment starting from A_2 . This produces two 30-60-90 right triangles, where the 60 degree side is of length 9. The hypotenuse, the length of a side of the hexagon, is therefore $6\sqrt{3}$.

10. At the ballroom dance competition, there were 10 couples. Each person shook hands with everyone except their partner. How many handshakes took place?

(a) 45

(b) 100

(c) 180

(d) 190

(e) 200

Solution: (C) If the partners were to shake each others hands, there would be $\binom{20}{2} = 190$ handshakes. Subtracting the 10 partner handshakes that did not occur leaves 180.

11. The average of three positive integers is 5. The average of their reciprocals is $17/72$. Their product is 96. What is the median of the three numbers?

(a) 3

- (b) 4
- (c) 5
- (d) 6
- (e) 7

Solution: (B) Guess and check works well here. What three integers have a product of 96 whose average is 5? Notice $96 = 3 \cdot 2^5$. The three numbers are 3, 4 and 8. This gives a median of 4, product of 96, and average of reciprocals $1/8 + 1/4 + 1/3 = 17/(24 \cdot 3)$ is $17/72$. More explicitly, consider the factors of 96. In order for the average of the three integers to be five, the largest factor we can use is 12, but this forces us to use (12, 2, 1) as the three integers, and their product is 24. So this leaves 1,2,3,4, 6, 8 as the choices for the integers. The only triple left that gives a product of 96 is (3, 4, 8).

12. A library shelf contains seven books. Three books are math books and four books are history books. In how many ways can the books be arranged on the shelf so that all the math books will be together?
- (a) 24
 - (b) 120
 - (c) 144
 - (d) 240
 - (e) 720

Solution: (E) There are five slots for the math books. We can order the math books in 6 different ways and can order the history books in 24 different ways. This gives $24 \cdot 5 \cdot 6 = 720$.

13. There are 29 people in a room. Of these 9 speak at least Spanish, 22 speak at least English and 5 speak neither Spanish or English. How many people in the room speak both Spanish and English?
- (a) 3
 - (b) 4
 - (c) 6
 - (d) 7
 - (e) 8

Solution: (D) Of the possible speakers of both Spanish and English, 22 of 24 speak English. So there are two speakers that speak only Spanish. The other seven Spanish speakers speak English.

14. How many regular polygons, with number of sides less than 100, have interior angle measures whose values are integer degrees?

- (a) 11
- (b) 16
- (c) 19
- (d) 20
- (e) 21

Solution: (C) Look at the factors of $360 = 2^3 3^2 5$. Each factor under 100 gives the number of sides of a regular polygon where the interior angle measures are values of integer degrees. This list is: 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90. There are 19 numbers on this list.

15. Suppose the first five terms of an arithmetic progression are $a, x, b, 2x, c$. What is the ratio of a to c ?

- (a) $1/5$
- (b) $1/4$
- (c) $2/5$
- (d) $3/5$
- (e) $4/5$

Solution: (A) The difference between successive terms is $x/2$. As a result $a = x/2, e = 5x/2$. So the ratio is $1/5$.

16. Consider 5 distinct points in the plane. If we draw line segments (possibly curves) between each pair of points, what is the minimum number of times these segments cross?

- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) 4

Solution: (B) This is equivalent to the *crossing number* of the complete graph on five vertices(in graph theory). By Kuratowski's theorem, the answer is not zero, and we can construct a drawing where there is only one crossing.

17. How many ordered pairs of integers (x, y) are solutions to

$$x^3 + y^4 = 2009?$$

- (a) 0
- (b) 1
- (c) 2
- (d) 3
- (e) 4

Solution: (A) Notice that 2009 is congruent to 7 modulo 13. The residues of a cube mod 13 are 0,1,5, 8 or 12, while the residues of a fourth power are 0,1,3 or 9. Notice then there are no integral solutions to our equation.

18. If the sum of the first $2n$ positive integers is 77 more than the sum of the first n positive integers, then the sum of the first $3n$ integers is:

- (a) 49
- (b) 105
- (c) 120
- (d) 210
- (e) 231

Solution: Let's solve for n :

$$\frac{n(n+1)}{2} + 77 = \frac{2n(2n+1)}{2}$$

This gives $n = 7$. The sum of the first 21 integers is 231.

19. If a hen and a half can lay an egg and a half in a day and a half, how many days will it take for 10 hens to lay 60 eggs?

- (a) 6
- (b) $7\frac{1}{2}$
- (c) 8
- (d) 9
- (e) 12

Solution: (D) By the problem statement, 1.5 hens can lay one egg in one day. So each hen lays $\frac{2}{3}$ of an egg each day. So 10 hens lay $\frac{20}{3}$ eggs per day. We need $\frac{180}{3}$ total eggs, so this takes $\frac{180}{20} = 9$ days.

20. Consider the regular hexagon $ABCDEF$. Let ACE be a triangle drawn inside the hexagon. What is the ratio of the area of ACE to the area of $ABCDEF$?

(a) $\frac{1}{6}$

(b) $\frac{1}{3}$

(c) $\frac{1}{2}$

(d) $\frac{2}{3}$

(e) $\frac{5}{6}$

Solution: (C) From the center of the hexagon draw a point P and draw line segments PA, PC, PE . From this construction it is clear to see the ratio is $1/2$.