Georgia Institute of Technology

High School Mathematics Competition 2008

Varsity Proof-Based Test
Problem #1

Is there any pair of positive integers a and b such that the number

$$(2a + b)(2b + a)$$

is a power of 2? Justify your answer.

Solution: Let 2^m be the maximum power of 2 dividing a and 2^n the maximum power of 2 dividing b. Given the symmetry of the problem, we may assume without any loss of generality that $m \leq n$. Call $\hat{a}$ and $\hat{b}$ the positive integer numbers such that $a = 2^m\hat{a}$ and $b = 2^m\hat{b}$. Note that $\hat{a}$ has to be an odd number. Now

$$(2a + b)(2b + a) = 2^m(2a + b)(2\hat{b} + \hat{a}),$$

But $2\hat{b} + \hat{a} \geq 3$ and it's an odd number, thus $(2a+b)(2b+a)$ has at least one odd prime factor, and by uniqueness of the prime factorization, such number cannot be a power of 2, regardless of what pair (a, b) is chosen.

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Problem #2

Let a, b, c and d distinct real numbers such that $a > 0$, the equation $x^2 - 3ax - 8b = 0$ has roots c and d, and the equation $x^2 - 3cx - 8d = 0$ has roots a and b. Find the value of the sum $a + b + c + d$.

Solution: Note that under these conditions, none of the numbers a, b, c or d can be 0. If $a = 0$ or $b = 0$, then 0 is a root of $x^2 - 3cx - 8d = 0$, and hence $d = 0$, but we know all the numbers are different, so, this cannot be the case. In the same way neither d nor c can be 0. Now, $x^2 - 3ax - 8b = (x - c)(x - d)$ and $x^2 - 3cx - 8d = (x - a)(x - b)$, hence

$$\begin{aligned} a + b &= 3c \qquad & ab &= -8d \\ c + d &= 3a \qquad & cd &= -8b \end{aligned}$$

Adding the two equations on the left, one has that $a + b + c + d = 3(a + c)$ or equivalently $b + d = 2(a + c)$. On the other hand, multiplying the two equations on the right one gets that $abcd = 64bd$, and as none of the numbers is zero, one can simplify to $ac = 64$. Now, if one evaluates the first polynomial in a and the second in c, one would get that

$$c^2 - 3ac - 8b = 0 \text{ and } a^2 - 3ac - 8d = 0.$$

Adding these two equalities, one gets

$$\begin{aligned} 0 &= (a^2 + c^2) - 6ac - 8(b + d) \\ &= (a^2 + 2ac + c^2) - 16(a + c) - 8ac \\ &= (a + c)^2 - 16(a + c) - 512 \\ &= [(a + c) + 16][(a + c) - 32] \end{aligned}$$

Note that as $a > 0$ and $ac = 64$, then c is also positive, and so is $a + c$. Therefore $a + c = 32$ and hence $a + b + c + d = 96$.

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Problem #3

Define

$$n!! = n(n-2)(n-4)(n-6)\cdots = \prod_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (n-2k).$$

Prove the identity

$$\frac{(2n-3)!!2^n}{n!} = \frac{1}{2n-1}\binom{2n}{n}$$

Solution:

$$\begin{aligned}
\frac{1}{2n-1}\binom{2n}{n} &= \frac{1}{2n-1} \cdot \frac{(2n)!}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!(2n)(2n-2)(2n-4)\cdots 2}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!2^n(n)(n-1)(n-2)\cdots 1}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!2^n(n!)}{(n!)(n!)} \\
&= \frac{1}{2n-1} \cdot \frac{(2n-1)!!2^n}{n!} \\
&= \frac{(2n-3)!!2^n}{n!}
\end{aligned}$$

$\square$

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Varsity Proof-Based Test
Problem #4

Prove that for every positive integer n, there is a polynomial $p_n(x)$ with integer coefficients and degree n such that for every real number θ, $\cos(n\theta) = P_n(\cos\theta)$.

Solution: For this proof we will strongly use the identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$. Note that

$$\begin{aligned}\cos[(n+1)\theta] &= \cos(n\theta)\cos\theta - \sin(n\theta)\sin\theta \\ \cos[(n-1)\theta] &= \cos(n\theta)\cos\theta + \sin(n\theta)\sin\theta.\end{aligned}$$

and adding these two identities one gets

$$\cos[(n+1)\theta] = 2\cos(n\theta)\cos\theta - \cos[(n-1)\theta].$$

Now, define $P_1(x) = x$, $P_2(x) = 2x^2 - 1$, and for $n \geq 2$ define $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$. By induction, as $P_1(x)$ and $P_2(x)$ have integer coefficients, and if $P_n(x)$ and $P_{n-1}(x)$ then $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$ has integer coefficients, then, what is left to prove is that for every n it holds that $\cos(n\theta) = P_n(\cos\theta)$.

Note that $P_1(\cos\theta) = \cos\theta$ and $P_2(\cos\theta) = 2\cos^2\theta - 1 = \cos 2\theta$. Now, by induction, assume that $P_n(\cos\theta) = \cos(n\theta)$ and that $P_{n-1}(\cos\theta) = \cos[(n-1)\theta]$. Then, as $P_{n+1}(x) = 2xP_n(x) - P_{n-1}(x)$, we have that

$$\begin{aligned}P_{n+1}(\cos\theta) &= 2\cos\theta P_n(\cos\theta) + P_{n-1}(\cos\theta) \\ &= 2\cos(n\theta)\cos\theta - \cos[(n-1)\theta] \\ &= \cos[(n+1)\theta].\end{aligned}$$

These are known as Chebyshev polynomials of the first kind.

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Varsity Proof-Based Test
Problem #5

Let A, B, C and D be four different points in the plane such that the angles $\sphericalangle ACD$ and $\sphericalangle BCD$ are obtuse. Prove that for any point E with $CE \leq CD$ the following inequality holds:

$$2AD \cdot BD > (AC + CD) \cdot BE.$$

Solution: Note that as $\sphericalangle ACD$ and $\sphericalangle BCD$ are obtuse, then

$$AD^2 > AC^2 + CD^2 \geq \frac{(AC+CD)^2}{2} \Rightarrow \sqrt{2}AD > (AC + CD),$$

$$BD^2 > BC^2 + CD^2 \geq \frac{(BC+CD)^2}{2} \Rightarrow \sqrt{2}BD > (BC + CD).$$

Multiplying these two inequalities, one gets that

$$2AD \cdot BD > (AC + CD) \cdot (BC + CD).$$

Now, as $CD \geq CE$, then $BC + CD \geq BC + CE \geq BE$ by triangle inequality, and thus

$$2AD \cdot BD > (AC + CD) \cdot BE.$$

$\square$