Georgia Tech HSMC 2007

Varsity Proof

Solutions

1. Show that if x, y, z, w are positive real numbers, then

$$\frac{(x^2+x+1)(y^2+y+1)(z^2+z+1)(w^2+w+1)}{xyzw} \geq 81.$$

Solution: First note that $(x-1)^2 \geq 0$ for any x, in particular if $x > 0$. Then $x^2+1 \geq 2x$, and therefore $(x^2+x+1) \geq 3x$. The same is true for all the other variables. Then multiplying all four

$$(x^2+x+1)(y^2+y+1)(z^2+z+1)(w^2+w+1) \geq 81xyzw$$

As all the variables are positive, we can divide by $xyzw$, and therefore

$$\frac{(x^2+x+1)(y^2+y+1)(z^2+z+1)(w^2+w+1)}{xyzw} \geq 81.$$

□

2. Find the number of paths (that is, moving only vertically or horizontally) in the following array which spell out the word $MATHEMATICIAN$.

```
            M
           MAM
          MATAM
         MATHTAM
        MATHEHTAM
       MATHEMEHTAM
      MATHEMAMEHTAM
     MATHEMATAMEHTAM
    MATHEMATITAMEHTAM
   MATHEMATICITAMEHTAM
  MATHEMATICICITAMEHTAM
 MATHEMATICIAICITAMEHTAM
MATHEMATICIANAICITAMEHTAM
```

Solution: Consider half of the problem as shown below and notice that all such path have to arrive to the same `N`.

```
            M
           MA
          MAT
         MATH
        MATHE
       MATHEM
      MATHEMA
     MATHEMAT
    MATHEMATI
   MATHEMATIC
  MATHEMATICI
 MATHEMATICIA
MATHEMATICIAN
```

Going backward, you can see that to go from one letter to the previous, there are always two posibilities. Then one has 2 posibilities to go from `N` to `A`, two from `A` to `I`, and so on. That give us 2^{12} posibilities. Now, on the other half we also have 2^{12} posibilities, and the only mutual path to both sets is the one that is directly vertical. Hence, there are $2^{13} - 1 = 8191$ different possible paths.

□

3. Show that in every tetrahedron, there must be at least one vertex at which each of the face angles is acute.

   Solution: First, note that the sum of all the face angles of a tetrahedron is 4π. Now, suppose that $ABCD$ is the tetrahedron. Note that, assuming $\sphericalangle BAC$ is the largest of the angles from the vertex A, then note that $\sphericalangle BAC < \sphericalangle BAD + \sphericalangle CAD$, and therefore $2\sphericalangle BAC < \sphericalangle BAC + \sphericalangle BAD + \sphericalangle CAD$, hence the sum of the angles at any given vertex is strictly larger than twice the largest angle. If every vertex has an angle of at least $\frac{\pi}{2}$, then the sum S of the angles of the tetrahedron would be $S > 4 \cdot 2 \cdot \frac{\pi}{2} = 4\pi$, and this is a contradiction, therefore there is at least one vertex with all its angles acute.

□

4. Prove that if α, β and γ are the angles of a triangle, then

$$\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$$

Solution: First, note that $\alpha + \beta + \gamma = \pi$, hence $\alpha + \beta = \pi - \gamma$, and therefore

$$\begin{aligned} -\tan\gamma &= \tan(\pi - \gamma) \\ &= \tan(\alpha + \beta) \\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \end{aligned}$$

and therefore

$$-\tan\gamma + \tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta$$

$$\Rightarrow \tan\alpha\tan\beta\tan\gamma = \tan\alpha + \tan\beta + \tan\gamma$$

□

5. The square numbers are numbers of the form n^2 for some n. The triangular numbers are numbers of the form $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for some n. Show that there are infinitely many numbers that are both square and triangular numbers.

Solution: First than all, let's see that we do have a initial case. 1 is both a triangular and a square number. Then, define $a_1 = 1$, $b_1 = 1$, and define recursively $a_{n+1} = 4a_n(a_n + 1)$ and $b_{n+1} = 2b_n(2a_n + 1)$. First note that for any n one has that $a_n < a_{n+1}$ and $b_n < b_{n+1}$, then, both sequence of positive integers are strictly increasing. Now, we will prove that for any n

$$\frac{a_n(a_n + 1)}{2} = b_n^2.$$

Note that this is true for $n = 1$. Now, suppose it is true for $n = k$, then, note that

$$\begin{aligned} \frac{a_{k+1}(a_{k+1} + 1)}{2} &= \frac{4a_k(a_k + 1)[4a_k(a_k + 1) + 1]}{2} \\ &= 4\frac{a_k(a_k + 1)}{2}(2a_k + 1)^2 \\ &= 4b_{k+1}^2(2a_k + 1)^2 \\ &= [2b_k(2a_k + 1)]^2 \\ &= b_{k+1}^2 \end{aligned}$$

Hence, we have two increasing infinite sequences a_n and b_n such that the triangular number of the first equals the square of the other, hence, there are infinitely many numbers that are both square and triangular.

□