Georgia Tech HSMC 2006

Junior Varsity Proof

Solutions

1. Show that at a gathering of any six people, some three of them are either mutual acquaintances or are complete extrangers to each other.

   Solution: Without loss of generality, assume this people are Alice, Brenda, Charles, David, Ellen and Frank, or A, B, C, D, E and F as a shortcut. From A's standpoint, she either is an acquaintance with at least three people, or is a totally extranger with at least three people. Suppose then, without loss of generality that A is acquaintance with B, C and D. Now, if among B, C and D, they are all complete extrangers to each other, then the statement is true. If not, then suppose that B and C are mutual acquaintances (symetrically any other pair), then A, B and C are mutual acquaintances, and hence, the statement is true.

   $\square$

2. A regular tetrahedron and a regular octahedron have equal edges. Find the ratio of their volumes.

   Solution: Let ℓ be the length of the edges. A tetrahedron is a *pyramid with triangular base*, and a octahedron can be seen as two *pyramid with square basis* glued together by the base. Consider the vertices of the tetrahedron as X, Y, Z and W, and let V be the the foot of the height of the tetrahedron over XYZ. Now, let A, B, C, D, E, F be the vertices of the octahedron, where $ABCD$ is the square basis. Let G be the foot of the heigt over the square base.

   The volume of all piramida figure with straight edges is the area of the base times the height divided by 3. Let h_1 be the height of the tetrahedron and h_2 the height of the pyramid $ABCDE$. By symetry XV bisects the angle $\sphericalangle YXZ$, YV bisects $\sphericalangle XYZ$, hence V is the circumcenter of $\triangle XYZ$, and therefore $XV = \frac{\sqrt{3}}{3}\ell$, and by the Pythagorean theorem one have that $h_1 = \frac{\sqrt{6}}{3}\ell$. Now, AG is half of the diagonal of $ABCD$, hence $AG = \frac{\sqrt{2}}{2}\ell$ and hence $h_2 = \frac{\sqrt{2}}{2}\ell$. Let $\triangle$ be the area of XYZ, and

$\square$ the area of $ABCD$. If Λ_1 is the volume of the tetrahedron and Λ_2 be the volume of the octahedron, then,

$$\Lambda_1 = \frac{h_1\triangle}{3} = \frac{\left(\frac{\sqrt{6}}{3}\ell\right)\left(\frac{\sqrt{3}}{4}\ell^2\right)}{3} = \frac{\sqrt{2}}{12}\ell^3$$

$$\Lambda_2 = 2\frac{h_2\square}{3} = \frac{2\left(\frac{\sqrt{2}}{2}\ell\right)\ell^2}{3} = \frac{\sqrt{2}}{3}\ell^3$$

Thus

$$\frac{\Lambda_1}{\Lambda_2} = \frac{1}{4}.$$

$\square$

3. Find the number of paths (that is, moving only vertically or horizontally) in the following array which spell out the word *GEORGIA*.

```
      G
     GEG
    GEOEG
   GEOROEG
  GEORGROEG
 GEORGIGROEG
GEORGIAIGROEG
```

Solution: Consider half of the problem as shown below and notice that all such path have to arrive to the same `A`.

```
      G
     GE
    GEO
   GEOR
  GEORG
 GEORGI
GEORGIA
```

Going backward, you can see that to go from one letter to the previous, there are always two posibilities. Then one has 2 posibilities to go from `A` to `I`, two from `I` to `G`, and so on. That give us 2^6 posibilities. Now, on the other half we also have 2^6 posibilities, and the only mutual path to both sets is the one that is directly vertical. Hence, there are $2^7 - 1 = 127$ different possible paths.

$\square$

4. Solve

$$\begin{cases} a^3 - b^3 - c^3 &= 3abc \\ a^2 &= 2(b+c) \end{cases}$$

simultaneously in the positive integers.

Solution: We will first work the first equation. Note that

$$\begin{aligned} 3abc &= a^3 - b^3 - c^3 \\ &= a^3 - (b^3 + c^3) \\ &= a^3 - (b+c)(b^2 - bc + c^2) \\ &= a^3 - (b+c)\left[(b+c)^2 - 3bc\right] \\ &= \left[a^3 - (b+c)^3\right] + 3bc(b+c) \\ &= [a - (b+c)]\left[a^2 + a(b+c) + (b+c)^2\right] + 3bc(b+c) \end{aligned}$$

therefore

$$\begin{aligned} 0 &= [a - (b+c)]\left[a^2 + a(b+c) + (b+c)^2\right] + 3bc(b+c) - 3abc \\ &= [a - (b+c)]\left[a^2 + a(b+c) + (b+c)^2\right] - 3bc\,[a - (b+c)] \\ &= [a - (b+c)]\left[a^2 + a(b+c) + (b+c)^2 - 3bc\right] \\ &= [a - (b+c)]\left[a^2 + a(b+c) + (b-c)^2 + bc\right] \end{aligned}$$

Nothe that the second member of the multiplication is the addition of strictly positive numbers, hence the second bracket is different from zero, and thus,

$$a - (b+c) = 0 \Rightarrow a = b + c = \frac{a^2}{2} \Rightarrow a = 2 \Rightarrow b = c = 1$$

What is the only solution.

$\square$

5. Prove that if a, b, c are real numbers such that $a + b + c = 0$, then

$$3abc = a^3 + b^3 + c^3.$$

SolutionNote that as $a+b+c=0$ then we have that

$$\begin{aligned}
0 &= (a+b+c)^3 \\
&= a^3+b^3+c^3+3ab^2+3ac^2+3ba^2 \\
&\quad +3bc^2+3ca^2+3cb^2+6abc \\
&= (a^3+b^3+c^3)-3abc+3(ab^2+ba^2+abc) \\
&\quad +3(cb^2+bc^2+abc)+3(ac^2+ca^2+abc) \\
&= (a^3+b^3+c^3)-3abc+3ab(a+b+c) \\
&\quad +3cb(a+b+c)+3ac(a+b+c) \\
&= (a^3+b^3+c^3)-3abc
\end{aligned}$$

From where evidently $3abc = a^3+b^3+c^3$

□