Georgia Tech HSMC 2006

Junior Varsity Proof

Solutions

1. The positive real numbers x, y satisfy $x+y=2$. Prove that $xy(x^2+y^2) \leq 2$

   Solution

$$\begin{array}{rrcl} & 0 & \leq & (xy-1)^2 \\ \Rightarrow & 0 & \leq & (xy)^2 - 2xy + 1 \\ \Rightarrow & 2xy - (xy)^2 & \leq & 1 \\ \Rightarrow & xy(2-xy) & \leq & 1 \\ \Rightarrow & xy(4-2xy) & \leq & 2 \\ \Rightarrow & xy[(x+y)^2 - 2xy] & \leq & 2 \\ \Rightarrow & xy(x^2+y^2) & \leq & 2 \end{array}$$

□

2. Among all pairs of 6-digit natural numbers a, b (in decimal representation) that have the same set of digits with all digits positive, what isthe largest possible value of $a-b$? Prove your results. For example, for 2-digit numbers the largest value is $91-19=72$.

   Solution: Note that if we give a set of digits $d_0,\ d_1,\ d_2,\ d_3,\ d_4,\ d_5$, assuming $1 \leq d_0 \leq d_1 \leq d_2 \leq d_3 \leq d_4 \leq d_5 \leq 9$, then, the greates value would be $a = d_5d_4d_3d_2d_1d_0$ and the smallest would be $b = d_0d_1d_2d_3d_4d_5$. Thus

$$\begin{aligned} a-b &= (10^5 d_5 + \cdots + 10d_1 + d_0) \\ &\quad -(10^5 d_0 + \cdots + 10d_4 + d_5) \\ &= (10^5-1)(d_5-d_0) \\ &\quad +(10^4-10)(d_4-d_1) \\ &\quad +(10^3-10^2)(d_3-d_2) \end{aligned}$$

   Therefore we are trying to maximize the numbers $d_5-d_0,\ d_4-d_1$ and d_3-d_2, what is obtained when $d_0=d_1=d_2=1$ and $d_3=d_4=d_5=9$, and therefore $a-b=887112$.

□

3. In a marching band practice, the members are arranged in a rectangular formation. Looking at the people in this formation, the director notes that among those who are the shortest in their respective columns, Al is the tallest. Also, among those who are the tallest in their respective rows, Bob is the shortest. Assuming that no 2 people have the same height, is Al taller than Bob?

   Solution If Bob and Al are either in the same column or in the same row, then Bob has to be taller than Al, because Bob is the tallest in his row, and Al is the shortest in his column. Now, if Al and Bob are in different rows and different columns, then, let's suppose the guy in the same column than All and the same row than Bob is named Tom. Then, by the definition, Tom is taller than Al and Tom is shortest than Bob, and by transitivity, Al is shortest than Bob in either case.

□

4. A popular game in Asia is to take four numbers and to try combining them into the number 24 using only $+, -, \times, \div$. Each number must be used once and only once. For example, with the four numbers $2, 4, 5, 8$ we can get 24 in a number of ways:

$$(4 \times 5) + (8 \div 2) = 24,$$

$$\text{or } (2 + 5 - 4) \times 8 = 24,$$

$$\text{or } 2 \times 4 \times (8 - 5) = 24.$$

Can you make 24 using the numbers $5, 5, 5, 1$?

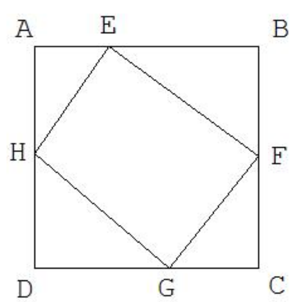
Solution: Yes, you can,

$$24 = 5[5 - (1 \div 5)]$$

□

5. Consider a square and a rhombus inscribed in the square with its vertices on the bounadry of the square. Prove that the rhombus is in fact a square.

Solution Consider the square to be $ABCD$, and the rhombus inscribed to be $EFGH$ with E in AB, F in BC, G in CD and H in DA as in the figure.



Now, as $AB \parallel DC$ and $EH \parallel FG$, then $\angle AEH = \angle FGC$. Similarly $\angle AHE = \angle CFG$. As $EFGH$ is a rhombus, then $EH = FG$, and therefore $\triangle AEH \cong \triangle CGF$, and therefore $GC = AE$ and $AH = CF$. Similarly $EB = DG$ and $BF = DH$.

Let's call $\ell = AB = BC = CD = DA$, $a = AE = CG$ and $b = BF = DH$. Notice that as $EFGH$ is a rhombus, then

$$\begin{aligned} EH^2 &= EF^2 \\ \Rightarrow \quad AH^2 + AE^2 &= EB^2 + BF^2 \\ \Rightarrow \quad a^2 + \ell^2 + b^2 - 2b\ell &= a^2 + \ell^2 + b^2 - 2b\ell \\ \Rightarrow \quad a &= b \end{aligned}$$

Therefore

$$\begin{aligned} \triangle HAE &\cong \triangle EBF \\ &\cong \triangle FCG \\ &\cong \triangle GDH \end{aligned}$$

and hence $\angle EHA + \angle GHD = \pi/2$, but $\angle EHA + \angle GHD + \angle EHG = \pi$, thus $\angle EHG = \pi/2$. Similarly $\angle FEH = \angle EFG = \angle HGF = \pi/2$, and therefore $EFGH$ is a square.

□