2005 Georgia Tech High School Mathematics Competition
Junior-Varsity Multiple-Choice Examination – Version A
SOLUTIONS

Problem 1: If $f(x) = \frac{x^4-3x^3+x^2-2}{x-3}$, what is the value of $f(4)$?

(A) 62 (B) 70 (C) 78 (D) 81 (E) 90

$$f(4) = \frac{4^4 - 3\cdot 4^3 + 4^2 - 2}{4-3} = 78$$ Answer is (C)

Problem 2: Evaluate the expression:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=0}^{3} i\,j$$

(A) 12 (B) 36 (C) 48 (D) 288 (E) None of the above

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=0}^{3} i\,j = \left(\sum_{i=1}^{3} i\right)\cdot\left(\sum_{j=1}^{3} j\right)\cdot\left(\sum_{k=0}^{3} 1\right) = 6\cdot 6\cdot 4 = 144$$ Answer is (E)

Problem 3: Express $\frac{1}{1-\sqrt{2}}$ in the form $\alpha+\beta\sqrt{2}$, where α and β are rational numbers.

(A) $1-\sqrt{2}$ (B) $-1-\sqrt{2}$ (C) $\frac{-1}{3}-\frac{\sqrt{2}}{3}$ (D) $\frac{1}{3}+\frac{\sqrt{2}}{3}$ (E) $\frac{1}{5}+\frac{\sqrt{2}}{5}$

Notice first of all that the quantity $\frac{1}{1-\sqrt{2}}$ is negative.

$$\frac{1}{1-\sqrt{2}}\cdot\frac{1+\sqrt{2}}{1+\sqrt{2}} = \frac{1+\sqrt{2}}{1-2} = -1-\sqrt{2}$$ Answer is (B)

Problem 4: A function is said to be **even** if $f(x) = f(-x)$ for all values of x. A function is said to be **odd** if $f(x) = -f(-x)$ for all values of x. The function $g(x) = f(x) - f(-x)$ is:

(A) Even (B) Odd (C) Even only when $x > 0$ (D) Neither even, nor odd

$$g(-x) = f(-x) - f(x) = (-1) \cdot (f(x) - f(-x)) = -g(x) \qquad g(x) \text{is odd; Answer is (B)}$$

Problem 5: Find the minimum value of $g(x) = 4x - x^2 + 1$, where x can be any real number.

$(A)\ -1$ (B) 1 (C) 5 (D) 7 (E) None of the above

$$g(x) \to (-\infty) \text{whenever} |x| \text{is large.} \qquad \text{Answer is (E)}$$

Problem 6: How many positive integer solutions are there to $1! + 2! + 3! + 4! + ... + x! \leq x^3$?

(A) 1 (B) 2 (C) 4 (D) 5 (E) None of the above

$$1! \leq 1^3 \qquad \sum_{k=1}^{2} k! = 3 \leq 8 = 2^3 \qquad \sum_{k=1}^{3} k! = 9 \leq 27 = 3^3 \qquad \sum_{k=1}^{4} k! = 33 \leq 64 = 4^3 \qquad \sum_{k=1}^{5} k! = 153 > 125 = 5^3$$

Answer is (C).

Problem 7: What is 342_5 written in base-10?

(A) 342 (B) 85 (C) 123 (D) 97 (E) None of the above

$$342_5 = 3 \cdot 5^2 + 4 \cdot 5^1 + 2 \cdot 5^0 = 75 + 20 + 2 = 97 \qquad \text{Answer is (D)}$$

Problem 8: What rational number can be represented by $0.\overline{099}$?

$(A)\ \frac{11}{101}$ $(B)\ \frac{11}{111}$ $(C)\ \frac{11}{121}$ $(D)\ \frac{11}{131}$ $(E)\ \frac{99}{1000}$

$$0.\overline{099} = \frac{99}{1000} \cdot \sum_{k=0}^{\infty} \frac{1}{1000} = \frac{99}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} = \frac{99}{999} = \frac{11}{111}$$

Answer is (B).

Problem 9: A triangle has vertices at $(8,6,6)$, $(8,12,6)$, and $(2,6,6)$. What is the area of this triangle?

(A) 9 (B) 12 (C) 18 (D) 27 (E) None of the above

Notice that this is a right triangle in the $z = 6$ plane. Therefore, area is $\frac{1}{2} \cdot (8-2) \cdot (12-6) = 18$. Answer is (C).

Problem 10: Let P be a point on the diagonal AC of the rectangle $ABCD$; how does the area of triangle APD compare to the area of triangle APB?

Area of Δ_{APD} ________ Area of Δ_{APB}

(A) $=$ (B) $>$ (C) $<$ (D) Not enough information given

Placing vertice A at the origin and C in the first quadrant, we may assume a rectangle having height H and length L; the point P may be placed anywhere along the diagonal, AC, which has the equation $y = \frac{H}{L}x$. Calculating the area of Δ_{APD}, we obtain $\frac{1}{2} \cdot (L-x) \cdot H$; calculating the area of Δ_{APB}, we obtain $\frac{1}{2} \cdot L \cdot (H - \frac{H}{L}x) = \frac{1}{2} \cdot (L-x) \cdot H$, which is identical. Answer is (A).

Problem 11: Given an isosceles right triangle inscribed in a circle where all three vertices of the triangle are located on the circle, determine the ratio of the area of the triangle to the area of the circle, $\frac{\text{Area}\Delta}{\text{Area}O}$.

(A) $\frac{1}{2\pi}$ (B) $\frac{2}{3\pi}$ (C) $\frac{1}{\pi}$ (D) $\frac{2}{\pi}$ (E) Not enough information given

Starting with the right angle corner, we always find that the other two intersections of the triangle with the circle create a hypotenuse that is a diameter of the circle, assumed to have any radius R. Then,

$$\frac{\text{Area}\Delta}{\text{Area}O} = \frac{\frac{1}{2}(\sqrt{2}R)^2}{\pi R^2} = \frac{1}{\pi}$$

Answer is (C).

Problem 12: The Fibonacci numbers are defined by the **recurrence relation**:

$$f_1 = 1 \qquad f_2 = 1 \qquad f_{n+2} = f_{n+1} + f_n$$

for $n = 1, 2, 3, \ldots$. The relation $f_n < 2^n$ is true for:

(A) Only the integers $n < 2^5$ (B) Only for integers $n < 2005$ (C) Only for integers $n < 2^{2005}$
(D) Only for integers $n < 2^{2005} - 2005$ (E) All positive integers n

Use induction: assume that the relation is true for some N steps (base cases work with initial conditions), and then show that the $(N+1)^{\text{st}}$ case holds.

$$f_{N+1} = f_N + f_{N-1} < 2^N + 2^{N-1} = \frac{3}{4}2^{N+1} < 2^{N+1}$$

Therefore the relation holds for all $n \geq 1$; Answer is (E).

Problem 13: How many words of length 8, formed using the alphabet (0,1,2,a,b) have at least one 'a' and one 'b'? For example, one such word is 01221aba.

(A) $5^8 - 4^8 - 4^8 + 3^8$ (B) $4^8 + 4^8 - 3^8$ (C) 3^8 (D) 4^8 (E) None of the above

The total number of length-8 words is 5^8; the number of length-8 words containing no A's is 4^8, and likewise the number of length-8 words containing no B's is also 4^8; and the number of length-8 words containing neither A's or B's is 3^8. Using inclusion and exclusion, the number of words containing at least one A and one B is

(Total Words) - (No A's) - (No B's) + (No A's or B's) $= 5^8 - 2 \cdot 4^8 + 3^8$ Answer is (A)

Problem 14: Let a, b, c, d be real positive numbers, with $\frac{a}{b} < \frac{c}{d}$. Which of the relations are true for:

$$\frac{a}{b} \;\underline{\qquad}\; \frac{a+c}{b+d} \;\underline{\qquad}\; \frac{c}{d}$$

(A) $<, <$ (B) $<, \leq$ (C) $\leq, <$ (D) $\leq, \leq$ (E) None of the above

Use cross multiplication and the given inequality, $\frac{a}{b} < \frac{c}{d}$, to obtain answer (A).

Problem 15: The number $2^{2005} + 2007$ can be written as the sum of two perfect squares in how many ways?

(A) 9 (B) 4 (C) 1 (D) 0 (E) None of the above

Take any number *modulo* 4 and square it – the result is 0 or 1 (0 if the number is evenly divisible by two and 1 otherwise). Therefore, for any number to be the sum of squares, it must be 0,1, or 2 *modulo* 4. $2^{2005} \equiv 0 \, mod \, 4$, so

$2^{2005} + 2007 \equiv 2007 \equiv 3\, mod\, 4$ cannot be the sum of two perfect squares. Answer is (D).

Problem 16: What digit occupies the $38,885^{\text{th}}$ position when we write out all the integers in succession, beginning with 1 (i.e. 1234567891011121314151617...)?

(A) 1 (B) 3 (C) 5 (D) 8 (E) 9

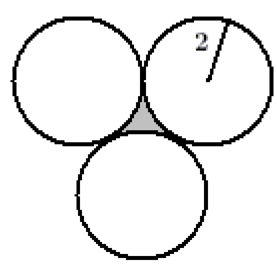
Counting the number of digits from 1-9 (9), 10-99 (2 times 90), 100-999 (3 times 900), and 1000-9999 (4 times 9000), we see that there will be $4 \cdot 9000 + 3 \cdot 900 + 2 \cdot 90 + 1 \cdot 9 = 38889$ digits in the sequence ending at 9999 (i.e. 12345.....999799989999). Counting backwards four digits, we find the digit 8. Answer is (D).

Problem 17: The fundamental theorem of algebra states that a polynomial of order n, $p_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + \ldots + \alpha_1 x + \alpha_0$, has exactly n roots. If all of the coefficients, $(\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0)$, are real numbers, which of the following statements are true? Mark all that apply.

(A) If n is even, there must exist at least one real root.
(B) If n is odd, there must exist at least one real root.
(C) If n is even, there must be at least one complex root.
(D) The sum of the coefficients, $\sum_{k=1}^{n-1} \alpha_k$, must be positive.
(E) All complex roots will occur in conjugate pairs.

Both (B) and (E) are true. A counterexample to (A) is $p_2(x) = x^2 + 1$, to (C) is $p_2(x) = x^2 - 1$, and to (D) is $p_1(x) = x - 1$.

Problem 18: Three mutually tangent circles of equal radius two are shown in the figure below. What is the area of shaded portion between the three circles?



$(A)\ \sqrt{3} - \frac{\pi}{2}$ $(B)\ \frac{4\sqrt{3} - \pi}{3}$ $(C)\ 4\sqrt{3} - 2\pi$

$(D)\ 2\sqrt{6} - \pi$ (E) Not enough information given

Mutually tangent circles create an equillateral triangle with their centers. Therefore, the area of the shaded region is the area of that triangle, $\frac{1}{2}\cdot 4\cdot(2\sqrt{3}) = 4\sqrt{3}$, minus the area of the circles inside the equillateral triangle, $3\cdot\frac{1}{6}\cdot\pi\cdot 2^2 = 2\pi$. Answer is (C).

Problem 19: Convergence of the sequence $X_n = (x_0, x_1, x_2, \ldots)$ implies that the quantity $|x_{n+1} - x_n|$ tends to zero in the limit as $n \to \infty$.

The recurrence relation, $z_n = \frac{3}{4}\cdot z_{n-1} + 3$, is a convergent sequence. Determine the value that it converges to.

(A) 3 $\quad(B)$ $\dfrac{9}{4}$ $\quad(C)$ 12 $\quad(D)$ 15 $\quad(E)$ None of the above

If Z_n is convergent, then $|z_{n+1} - z_n|$ tends to zero for sufficiently large n. Therefore, we rewrite the recurrence relation as $z_n = \frac{1}{4}z_n + \frac{3}{4}z_n = \frac{3}{4}z_{n-1} + 3$. Subtracting $\frac{3}{4}z_{n-1}$ from both sides, we obtain

$$\frac{1}{4}z_n + \frac{3}{4}(z_n - z_{n-1}) \quad\rightarrow\quad \frac{1}{4}z_n = 3$$

Therefore, the sequence Z_n converges to $\frac{3}{1/4} = 12$. Answer is (C).

Problem 20: The geometric sum, $1 + r + r^2 + \ldots + r^n + \ldots = \sum_{k=0}^{\infty} r^k$ converges to the value $\frac{1}{1-r}$ provided $|r| < 1$.

$$\text{Calculate} \qquad \sum_{k=1}^{\infty} \frac{6 - 2^{k+1}}{3^{k-1}}$$

(A) -13 $\quad(B)$ -12 $\quad(C)$ -3 $\quad(D)$ 6 $\quad(E)$ None of the above

$$\sum_{k=1}^{\infty} \frac{6 - 2^{k+1}}{3^{k-1}} = 6\cdot\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - 4\cdot\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k = 6\cdot\frac{1}{1-\frac{1}{3}} - 4\cdot\frac{1}{1-\frac{2}{3}} = -3$$

Answer is (C).