2004 Georgia Tech High School Mathematics Competition

Varsity Multiple Choice – Version A

1. When $x^{10}+1$ is divided by x^2-1, the remainder is

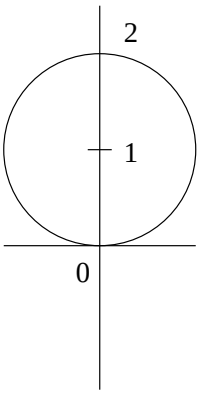
   (a) 1 (b) -1 (c) 0 (d) 2

   (e) none of these holds for every x.

2. Let $z = x + iy = re^{i\theta}$, which of the following equations corresponds to the circle graphed below?

   (a) $r = 1 + \sin\theta$ (b) $r = 2 + \sin\theta$ (c) $r = 2\cos\theta$

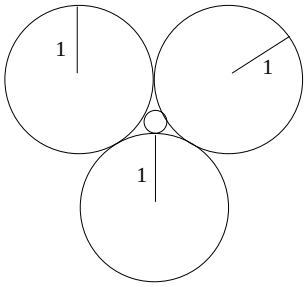
   (d) $r = 2\sin\theta$ (e) $r = 2 - \cos\theta$



3. The radius of the small circle in the middle is

   (a) $\frac{1}{6}$ (b) $\frac{1}{8}$ (c) $\frac{1}{4\sqrt{3}}$ (d) $\frac{\sqrt{3}-1}{4}$

   (e) $\frac{2\sqrt{3}-3}{3}$ (f) $\frac{2\sqrt{3}+1}{8}$ (g) $\frac{2\sqrt{3}}{9}$



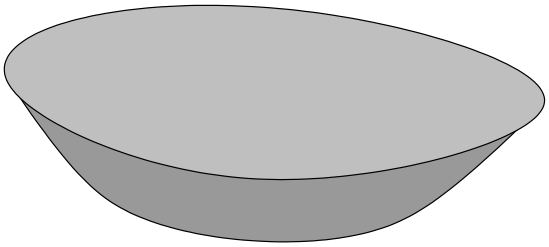
4. A drawer has 6 red socks and 6 white socks. If you reach in the drawer and randomly take out two socks, what is the chance (i.e., probability) that the two socks will match in color?

   (a) $\frac{5}{12}$ (b) $\frac{1}{2}$ (c) $\frac{2}{5}$ (d) $\frac{3}{7}$ (e) $\frac{5}{11}$ (f) $\frac{3}{8}$

5. A 1/50 scale model of a pond is shown below. If the volume of the scale model is 4 cm^3, then the volume of the actual pond is

   (a) 1/2 m^3 (b) 50000 cm^3 (c) 500 m^3

   (d) not determined by the given information (e) none of the above



6. A function $f : X \to Y$ is called one-to-one if for $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$. Define

$$\begin{aligned} f &: [-2, 0] \to \mathbb{R}, & f(x) &= \frac{x+1}{x-1} \\ g &: [0, 3] \to \mathbb{R}, & g(x) &= (x+1)^2 \\ h &: [-2, 1] \to \mathbb{R}, & h(x) &= \sqrt{x^2+1} \\ k &: [0, 3] \to \mathbb{R}, & k(x) &= (1-\sqrt{x})^2 \end{aligned}$$

   which of the following is true.

   (a) f and g are one-to-one functions

   (b) g, h, k are one-to-one functions

   (c) f, h, k are one-to-one functions

   (d) f, k are one-to-one functions

   (e) f, g, h, k are one-to-one functions

7. A number system based on 26 is used and the letters of the alphabet are the digits: $A = 0$, $B = 1$, $C = 2$, $D = 3, \cdots$, $X = 23$, $Y = 24$, $Z = 25$. In this system, how much is ONE + ONE?

   (a) DBIA (b) BAID (c) BDAI (d) DAIB (e) BDIA

8. Which of the following is logically equivalent to (P or Q) and (Q or R)?

   (a) P and R (b) (P and R) or Q (c) (P and Q) or (Q and R)

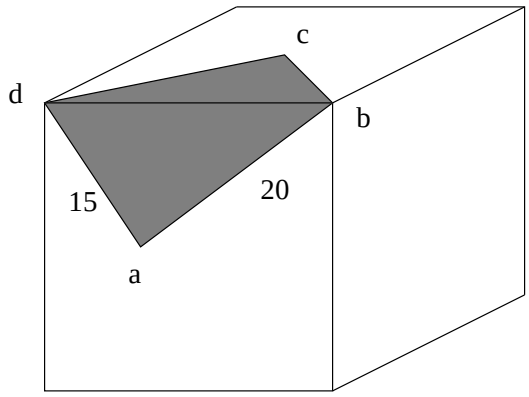
   (d) (Not Q) and (P or R) (e) None of the above

9. How many distinct solutions x are there for the equation $\det(A - xI) = 0$ where det stands for determinant, I is the identity matrix, and A is the matrix

$$\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$

   (a) no solution (b) 1 (c) 2 (d) 3 (e) more than 3

10. If a 15 by 20 rectangular piece of paper with corners a, b, c, and d is glued to the surface of a large cube so that the edge of the cube lies along the diagonal of the paper as shown, how far is corner a from corner c as measured through the cube?



   (a) $13\sqrt{2}$ (b) 18 (c) 18.2 (d) $\sqrt{337}$ (e) None of the above

11. If C is a circle with center $\left(\sqrt{2}, \sqrt{3}\right)$, what is the maximum number of points (m, n) with integer coordinates that lie on C?

    (a) 1 (b) 2 (c) 6 (d) 12 (e) no maximum

12. Assume that $\sin x + \cos x = a$. Find $\sin^4 x + \cos^4 x$ in terms of a.

    (a) a^4 (b) a^2 (c) $a^4 + a^2 + 1$ (d) $(1 + 2a^2 - a^4)/2$

    (e) none of the above

13. Solve $|3x^2 - 2x - 1| \geq 0$

    (a) $x \geq 1$ (b) $x \leq -1/3$ (c) $x \leq -1$ or $x \geq 1$

    (d) $1 \leq x \leq -1/3$ (e) none of the above

14. For $-\frac{\pi}{4} < x < 0$,

    (I) $\sin x < x$ (II) $\sin x > x$ (III) $\tan x < x$

    (IV) $\tan x > x$ (V) $\sin x > \cos x$ (VI) $\sin x < \cos x$

    which of the following is true:

    (a) (II, III, VI) (b) (II, IV, VI) (c) (I, III, VI)

    (d) (I, IV, VI) (e) (I, III, V)

15. Three boys A, B, and C are making a batch of cookies. It takes A working alone twice as long as it takes B working alone to make the same batch of cookies. It takes A four times as long as B and C working together to make the same batch. And it takes A 2 hours longer to make the batch than it takes all three boys working together. How long will it take A and C to make the same batch of cookies?

    (a) 1 hr. (b) $1\frac{1}{2}$ hr. (c) $\frac{1}{6}$ hr.

    (d) $\frac{5}{6}$ hr. (e) $\frac{3}{4}$ hr.

16. If $11^{100} - 1 = 10^n \cdot m$ for integers m and n with m not divisible by 10, then what is n?

    (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

17. Find the quadratic equation whose roots are the cubes of the roots of $x^2 + ax + b = 0$.

(a) $x^2 + a^3x + b^3 = 0$ (b) $ax^2 + bx + ab = 0$

(c) $x^2 + (a^3 + b^3)x + ab = 0$ (d) $x^2 + a(a^2 - 3b)x + b^3 = 0$

(e) None of the above

18. A plane can be divided into 2 regions by one line, 4 regions by two lines, and 7 regions by three lines. What is the maximal number of regions that a plane can be divided into by 100 lines?

(a) 451 (b) 851 (c) 1051 (d) 3051

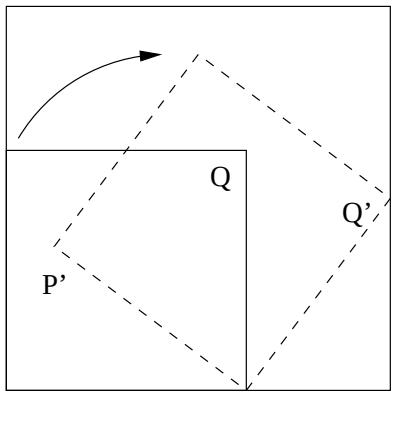
(e) 4051 (f) 5051 (g) 6051 (h) 7051

19. The equation $\sqrt{x+5} - \sqrt{x-2} + 1 = 0$ has

(a) one real root (b) one real root and 1 complex root

(c) 2 complex roots (d) two real roots (e) no root

20. A square of side length 5 rolls around the inside perimeter of a square of side length 8 as indicated in the figure. Eventually the corners P and Q will return to their original positions. If the first time this happens P has traveled a distance d, how far has Q traveled?



(a) $\frac{d-10\sqrt{2}}{1+\sqrt{2}}\,\pi$ (b) $10\tan^{-1}(3/4)$ (c) $d/(2\sqrt{2})$

(d) $10\pi + (\sqrt{2}+1)(10\pi - d)$ (e) None of the above