Georgia Tech High School Math Competition

Group Test

February 28, 2015

Write your names and answers on the provided answer sheet. You may collaborate only within your group on this test. Do not use outside notes or electronic devices. This test has **two** pages.

1. What is the maximal number of $1 \times 1 \times 4$ bricks you can fit in a $6 \times 6 \times 6$ box without intersecting the bricks or leaving the boundaries of the box?

Solution: Answer: 52. You can fit 52 bricks in a number of ways. To show we cannot fit 53 we can color the 27 $2 \times 2 \times 2$ sub-boxes that form the $6 \times 6 \times 6$ box in a check board pattern. Then, assuming we have colored the bottom left $2 \times 2 \times 2$ sub-box black, there will be 15 black and 14 white sub-boxes. But every $1 \times 1 \times 4$ brick placed inside the big box covers exactly 2 white and 2 black cells. Thus after placing 52 bricks in any possible way, we will have 8 black cells left, which can not be covered.

2. There are 28 passengers in a line, waiting to board a bus with exactly 28 seats. Tom is last in the line. Every passenger except the first one knows their seat number. When boarding, the first passenger chooses a seat at random. Every passenger after the first one sits in their designated seat, unless it is occupied at which point they choose a seat at random. What is the probability that Tom will sit in his designated seat.

Solution: Answer: $\frac{1}{2}$. Let us call the seat that the first passenger was supposed to sit in seat 1 and the one Tom was supposed to sit in seat 28. If seat 1 is occupied before seat 28, Tom will sit in his own seat. Otherwise he will not. The probability of this occurring is $\frac{1}{2}$ due to symmetry.

3. Adam, Steve, Greg, and John want to cross an old bridge at night. To be safe, they decide to cross at most two people at a time. However they only have one working flashlight, and since everyone crossing the bridge needs light, they can only cross at the speed of the slower person in a pair. If Adam, Steve, Greg, and John can cross in $2, 28, 20$, and 1.5 minutes respectively, what is the shortest time it would take all of them to cross the bridge?

Solution: Answer: 35.5. The way to do this is the following: First Adam and John cross which takes them 2 minutes, then John goes back with the flashlight, adding another 1.5 minutes, then Steve and Greg cross for 28 minutes and give the flashlight to Adam who goes back for 2 minutes and finally Adam and John cross again for another 2 minutes giving a total of 35.5.

4. 28 pirates found a chest with 2015 gold coins in it. The pirates agreed on the following method of dividing the fortune:
The most senior pirate suggests a distribution of the coins. Then they vote and if at least 50% of them agree that the distribution is fair, the gold is divided as proposed. Otherwise, the pirate who suggested the distribution of the coins is fed to the sharks and a new distribution is proposed by the second most senior pirate, etc.
Assume that all pirates are perfectly logical, and would abide by the following principles in order of importance:

   1. They want to survive;
   2. they want to get as much gold as possible;
   3. if neither of the two principles above are violated, they would rather feed their fellow pirates to the sharks.

   Assuming each pirate can only receive a whole number of coins and that no two pirates are of the same seniority, what will be the distribution of the coins?

Solution: Answer: If we number the pirates $1, 2, \ldots, 28$ in order of seniority, pirate 1 will receive 2002 coins and pirates $3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27$ will receive one coin each. We will prove the following statement by induction: pirates $3, 5, 7, \ldots$ receive one coin each and pirate 1 will get everything else. Indeed, if there is only one pirate he gets everything. If there are two pirates, 1 gets everything again (as he will vote for that distribution ensuring 50% of the votes). Assume the statement is true for some n. Now for $n + 1$, the most senior pirate knows that if his distribution is rejected, by the induction hypothesis, pirates $3, 5, \ldots$ will receive nothing. Thus giving one coin to each of them will win their vote and ensure his distribution is accepted. This ends the induction.

5. On the day of the competition, unexpectedly, snow started falling over Atlanta some time before noon, at a constant rate. Georgia Tech sent out a snow plow at noon to clean the campus. The snow plow removes snow at constant rate. If 2 miles of roads were cleared by 2PM and 4 miles of roads were cleared by 5PM, at what time did the snow start falling?

Solution: Answer: 8AM. Assume the snowfall started T hours before noon. If $A(t)$ is the amount of snow t hours after noon, we know that $A(t) = c(T + t)$ for some constant rate c. Furthermore, we know that the plow moves with speed inversely proportional to the amount of snow, i.e. $v = \frac{d}{A(t)} = \frac{d}{c(T+t)} = \frac{C}{t+T}$ where d is some constant and $C = \frac{d}{c}$. Now from the information in the question we know that:

$$2 = \int_0^2 \frac{C}{t+T} dt = C \log(2+T) - C \log T = C \log \frac{2+T}{T}$$

$$4 = \int_0^5 \frac{C}{t+T} dt = C \log(5+T) - C \log T = C \log \frac{5+T}{T}$$

So $\log \frac{5+T}{T} = 2 \log \frac{2+T}{T}$ and thus $\frac{5+T}{T} = (\frac{2+T}{T})^2$. So $(5+T)T = (2+T)^2$, i.e. $T = 4$. Therefore the snowfall started at $12 - T = 8$AM.

6. A Fermi Question is a question where you need to make educated guesses in order to estimate the order of magnitude of a certain quantity. What follows is a Fermi question. How many times is the volume of the Sun bigger then that of a Hindenburg class airship? You need to give your answer as the closest smaller power of 10, i.e. if your answer is 999 then you should write 10^2 and if it is 1000 you should write 10^3.
Hindenburg class airships were the largest aircraft ever to fly with length of 245 *meters.*

Solution: Answer: 10^{21}. The volume of the Sun is approximately $1.41 \cdot 10^{27}$ cubic meters and the volume of a Hindenburg class airship is approximately $2 \cdot 10^5$ cubic meters.