Georgia Tech High School Math Competition

Group Test

February 15, 2014

Write your names and answers on the provided answer sheet. You may collaborate only within your group on this test. Do not use outside notes or electronic devices. This test has **two** pages.

1. (2 points) Let $f(x)$ be a quadratic polynomial which satisfies the equation $30 + f(2x) = 4f(x) - 2x$ for all values of x. If $f(1) = 13$, what is the product of the roots of f?

Solution: Let $f(x) = ax^2 + bx + c$. Plugging this into the equation on f we get $30 + 4ax^2 + 2bx + c = 4ax^2 + 4bx + 4c - 2x$, which simplifies to $2x + 30 = 2bx + 3c$. Since this equation must be satisfied for all values of x, we must have $c = 10$ and $b = 1$. We also have that $f(1) = a + b + c = 13$ so then $a = 2$. The product of the roots of f is $c/a = 5$.

2. (2 points) Let ΔABC be a triangle with $AB = AC$, let X be a point on AB with $AX > XB$, and let D be on the extension of BC such that $XC = XD$. If the area of ΔABC is 7 times the area of ΔDXB, find the ratio AX/XB.

Solution: Write $r := AX/XB$. With some simple angle tracing we have $\angle ACX = \angle BXC - \angle BAC = \angle DXB$, so $\frac{[\Delta AXC]}{[\Delta DXB]} = \frac{(AC \cdot XC \cdot \cos \angle ACX)/2}{(XB \cdot XD \cdot \cos \angle DXB)/2} = \frac{AC}{XB} = r + 1$, together with $\frac{[\Delta ABC]}{[\Delta AXC]} = \frac{r+1}{r}$ we have $7 = \frac{(r+1)^2}{r}$, solving gives $r = \frac{5+\sqrt{21}}{2}$.

3. (2 points) Find all pairs of non-negative integers (n, m) such that

$$\binom{n}{m} = 3003$$

(where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ denotes a binomial coefficient).

Solution: Multiply $3003 = 3 \times 7 \times 11 \times 13$ by some $k!$ and see if $3003 \cdot k!$ can be factorized into k consecutive integers. After some computation we have $3003 = \binom{3003}{1} = \binom{78}{2} = \binom{15}{5} = \binom{14}{6}$, together with their reflection we have 8 representations.

The trial and error can stop here as for $k > 6$ either $\binom{n}{k} > \binom{n}{6} > \binom{14}{6} = 3003$ if $n \geq 2k$, or $\binom{n}{k} = \binom{n}{n-k}$ which is something we have considered. (This is the only known integer that can be represented in at least 8 ways using binomial coefficients.)

4. (2 points) Let A, B, C, D represent four distinct decimal digits such that the following equation holds:

$$\begin{array}{r} A\,B\,C \\ \times \quad C\,B\,A \\ \hline A\,D\,A\,B\,B\,C. \end{array}$$

Find the values of A, B, C, D.

Solution: $A = 1$, $B = 5$, $C = 6$, $D = 0$.

5. (3 points) Let S be a sphere with radius 1. Find the volume of each of the following polyhedra inscribed in S:

   a. a regular octahedron.

   b. a cube.

   c. a right prism with its face an equilateral triangle of side length 1.

Solution: (a) The octahedron can be considered as two square pyramids attached at their bases. The center of S is at the center of the square base. The height of each pyramid is a radius of S so has length 1, and the base is a square with side length $\sqrt{2}$ so the area is 2. The volume of each pyramid is $2/3$ so the total is $4/3$.

(b) Suppose the edge length of the cube is s. The distance from the center of the cube to a vertex is 1, so the the distance from a vertex to the opposite vertex is $2 = s\sqrt{3}$. Therefore $s = 2/\sqrt{3}$ and the volume is $s^3 = 8\sqrt{3}/9$.

(c) The area of the triangular face is $\sqrt{3}/4$, so we need only find the height h of the prism. A vertex of the triangle is at distance $\sqrt{3}/3$ from the center of the triangle. Taking an appropriate slice, we see that a radius to the vertex is at an angle of $\sin^{-1}(\sqrt{3}/3)$ from the direction of the height, and so the height is $h = 2\cos(\sin^{-1}(\sqrt{3}/3)) = 2\sqrt{1 - 1/3} = 2\sqrt{6}/3$. The volume is $\sqrt{2}/2$.

6. (2 points) Let S be the set of integers $\{2, 3, \ldots, N\}$ for some $N \geq 2$, and suppose that S can be partitioned into two disjoint subsets A and B with the following properties:

   - For any two integers x and y in A (not necessarily distinct), xy is not in A.
   - For any two integers x and y in B (not necessarily distinct), xy is not in B

   Find the maximum possible value of N.

   Solution: $N = 31$. To show that $N \geq 31$ we construct sets $A = \{2, 3, 16, \ldots, 31\}$ and $B = \{4, \ldots, 15\}$. For any x and y in B, the product $xy \geq 16$ so is not in B. For x and y in A, if $x \geq 16$ or $y \geq 16$, then $xy \geq 32$ which is not in A. If both are drawn from $\{2, 3\}$, then the possible values of xy are $4, 6, 9$ which are not in A.

   To show that $N < 32$. Let 2 be in A WOLOG. Then 4 must be in B, so 16 is in A, and then 8 must be in B. However $2 \cdot 16 = 4 \cdot 8 = 32$ so 32 cannot be in either set.

7. (2 points) Consider a 13×13 chess board with bottom-left square $(0, 0)$ and top-left square $(12, 12)$. Suppose a knight is sitting at $(0, 0)$.

   a. In how many different ways can the knight reach the top-right square, $(12, 12)$, in exactly 8 moves?

   b. Answer part a, but for a 12×12 chess board instead.

   (One knight move consists of moving 2 spaces in any of the four directions, and then 1 space in a perpendicular direction.)

   Solution: (a) Each knight move can be represented by a vector of the form $(\pm 1, \pm 2)$ or $(\pm 2, \pm 1)$. We want to find all sequences of eight such moves which sum up to $(13, 13) - (1, 1) = (12, 12)$. Ignoring the order of the moves, this can only be done in one way:

   $$4 \cdot (1, 2) + 4 \cdot (2, 1) = (12, 12).$$

   However we can order these moves anyway we like, which gives $\binom{8}{4} = 70$ ways.

   (b) To get $(11, 11)$ in eight moves, we need to double back one square, so there are two possible combinations of moves depending if we decide to go left or down:

   $$2 \cdot (1, 2) + 5 \cdot (2, 1) + (-1, 2) = (11, 11),$$

   $$5 \cdot (1, 2) + 2 \cdot (2, 1) + (2, -1) = (11, 11).$$

   For the first combination, we can order these eight moves in any way we want so long as we don't ever go off the side of the board. This is accomplished as long as $(-1, 2)$

does not occur first or last. Therefore there are 6 positions available for $(-1, 2)$, and then the other seven moves can be filled in any order, so $\binom{7}{2}$ ways. We get the same number of ways to form the other combination, so the total is

$$6 \cdot \binom{7}{2} + 6 \cdot \binom{7}{2} = 252.$$