2018 Georgia Tech High School Mathematics Competition
Group Test
10 March 2018

Instructions: This exam consists of six questions. You will have 90 minutes to complete as much of the exam as possible. You may collaborate only within your group on this test. Do not use outside notes or electronic devices. Write your names and answers on the provided answer sheet.

1. Compute $3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{\ldots}}}}}}$ and $3\sqrt[3]{3\sqrt[3]{3\sqrt[3]{3\sqrt[3]{3\sqrt[3]{3\sqrt[3]{\ldots}}}}}}$.

Solution: 9 and $3\sqrt{3}$.
Note that

$$
\begin{aligned}
3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{\ldots}}}}}} &= 3\left(3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{\ldots}}}}}}\right)^{1/2} \\
&= 3 \cdot 3^{1/2}\left(3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{3\sqrt{\ldots}}}}}}\right)^{1/4} \\
&= 3 \cdot 3^{1/2} \cdot 3^{1/4} \cdot \ldots \\
&= \prod_{i=0}^{\infty} 3^{1/2^i} \\
&= 3^{\sum_{i=0}^{\infty} 1/2^i} = 3^2 = 9.
\end{aligned}
$$

By the same argument, the later term evaluates to

$$3^{\sum_{i=0}^{\infty} 1/3^i} = 3^{\frac{1}{1-1/3}} = 3^{3/2} = 3\sqrt{3}.$$

Alternately, one can solve $3x = x^2$ and $3x = x^3$ for x, with $3x$ giving the final answer in both cases.

2. An ant is put on point A of a cube $ABCDA'B'C'D'$. The ant begins walking at a constant speed of 1 side per minute. In how many ways can the ant move to point D' (the farthest from A) in 9 minutes?

Solution: 4920.

We can split the vertices of the cube into three parts based on the distance from A, i.e., $a = \{A\}$, $b = \{B, C, A'\}$, $c = \{B', C', D'\}$ and $d = \{D'\}$. Let a_k, b_k, c_k, d_k be the number of ways to get to a vertex in a, b, c, d in k steps. We want to find d_9. We get the following system of equations:

$$\begin{array}{rcl} d_9 & = & c_8 \\ c_8 & = & 2b_7 + 3c_6 \\ b_7 & = & 2c_6 + 3b_5 \\ c_6 & = & 2b_5 + 3c_4 \\ b_5 & = & 2c_4 + 3b_3 \\ c_4 & = & 2b_3 + 3c_2 \\ b_3 & = & 2c_2 + 3b_1 \end{array}$$

Note that $c_2 = 6$ and $b_1 = 3$. Solving the system yields $d_9 = 4920$.

3. Let ABC be a right triangle with $\angle ACB = 90°$, CH be the height and CM - the median. Let N on MH be such that $MN = NC$ an $NH = HB + BC$.

   (a) Find $\angle BAC$.

   (b) Let K be on MC such that $KB \perp NC$. Find $\angle KNC$.

Solution: 10 and 10.

(a) Let D on NH be such that $ND = BC$. Then $DH = HB$ and $DNC \cong BHC$. Then $DC = BC = ND$ and $CM = AM = BM$. If $\angle BAC = \alpha$, then $\angle CMB = 2\alpha$. Then $\angle MCN = 2\alpha$ and $\angle CNB = 4\alpha$, $\angle NCD = 4\alpha$, and $\angle CDB = 8\alpha$. Therefore $\angle ABC = 8\alpha$ and $\alpha = 10$.

(b) Let L on NH be such that $\angle CBL = 60$. Then BCL is equilateral. Then $\angle KLC = \angle KCL = \angle NMC$ and so $KLMN$ can be inscribed in a circle. Note that $BLM \cong CLM$ and therefore $\angle KNC = \angle LMC = \frac{1}{2}\angle BMC = 10$.

4. Let $n \in \mathbb{N}$, and assume $n \geq 2$. For how many n is $\log_n(n+1)$ a rational number?

Solution: 0.

$\log_n(n+1)$ is not a rational number. Indeed, suppose $\log_n(n+1) = p/q$ for some $p, q \in \mathbb{Z}$. Since $n+1 > n$, we know that $p/q > 1$ and hence we may assume that $p, q > 0$. Then $n^{p/q} = n+1$, so $n^p = (n+1)^q$, where p, q are positive integers. But for such p and q, we know that n^p has the same parity as n, and $(n+1)^q$ has the same parity as $n+1$. Since n and $n+1$ are of opposite parity, it follows that $n^p \neq (n+1)^q$. By contradiction, we conclude that $\log_n(n+1)$ is not rational.

5. Find all natural numbers a and b such that $a^4 + a^2b^2 + b^4$ has a single prime factor, i.e., can be written as p^n where p is prime and $n \geq 1$.

Solution: Answer: $a = b = 3^k$ for $k \geq 0$. Since $a^4 + a^2b^2 + b^4 = (a^2 - ab + b^2)(a^2 + ab + b^2$, then $a^2 - ab + b^2 = p^m$ and $a^2 + ab + b^2 = p^n$, where p is prime and $0 \leq m \leq n \in \mathbb{Z}$. On the other hand, $3(a^2 - ab + b^2) \geq (a^2 + ab + b^2)$, so $m = n - 1$, $p = 3$ and $a = b = 3^k$ for $0 \leq k \in \mathbb{Z}$ or $m = n - 1$, $p = 2$ and $a^2 + b^2 = 3ab$ which is impossible.

6. You've just finished writing a proof for the Georgia Tech High School Math Competition, however, you're not sure if it is correct and you'd like to find out. Before you are three Georgia Tech students volunteering for the event, they are as follows:

   - The first one only likes seeing a correct proof and will always tell the truth.
   - The second one is very sadistic and only like seeing an incorrect proof (enjoying watching the students as they struggle) and always lies.
   - The third one is having a bad headache and so had the same inability today of determining if your proof is correct, but he'll like any proof you show him since he enjoys knowing that students are putting forth an effort. He also always tell the truth. If you ask him a yes/no something that requires knowing the validity of some proof, he will randomly say yes or no with probability 1/2.

   Unfortunately, you don't know which of the three volunteers are which. However, the three volunteers are all familiar with each other (including their lying habits, preference for proof correctness, and ability to determine the correctness of your proof).

   Of the three volunteers, one of them is asked to help grade some other problems leaving only two volunteers left. You are allowed to ask a single yes/no question and the two remaining volunteers will reply accordingly. What question should you ask to determine if your proof is correct?

   Solution: "Do you like my proof?" (Other answers may be possible)
   First, let's suppose that your proof is correct and look at what each volunteer would say.
   Volunteer 1: Since your proof is correct, she will like it. Since she tells the truth, she will say "yes" to liking it.
   Volunteer 2: Since your proof is correct, he doesn't like it. But he lies so he will say "yes" to liking it.
   Volunteer 3: Since he likes all proofs and tells the truth, he will say "yes".

   Now let's suppose that your proof is incorrect and look at what each volunteer would say.
   Volunteer 1: Since your proof is incorrect, she will not like it. Since she tells

the truth, she will say “no” to liking it.

Volunteer 2: Since your proof is incorrect, he likes it. But he lies so he will say “no” to liking it.

Volunteer 3: Since he likes all proofs and tells the truth, he will say “yes”.

We claim that you can determine if your proof is correct by looking at the number of “yes” responses you get. In particular, we claim that your proof is correct if and only if you get two “yes” responses.

In the case that your proof is correct, all three volunteers would have said “yes” so any two out of the three would “yes”.

In the case that your proof is incorrect, no matter which volunteer leaves, you can only have at most volunteer that says “yes”.

End of exam.

Answer Sheet.

Team Members:

Question 1:

Question 2:

Question 3, part 1:

Question 3, part 2:

Question 4:

Question 5:

Question 6: