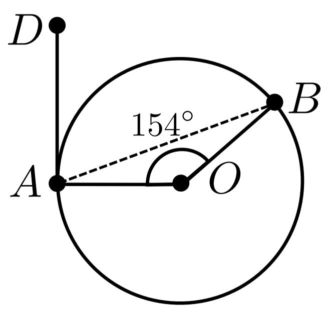
1 Fillins

1. Find the last 4 digits of 3333×6666.

Solution: 7778. By direct calculation, or observe that $3333 \times 6666 = 9999 \times 2222 = (10000 - 1) \times 2222 = 22220000 - 2222 = 22217778$.

2. How many ways are there to arrange 6 people in a circle? Here two arrangements are considered the same if each person is sitting directly to the left of the same person in both arrangements.

Solution: 120. There are 6! ways to arrange 6 people in a line. Each of the 6 positions of the circle is the starting point of a unique line. So the answer is $\frac{6!}{6} = 5! = 120$.



3. Let A and B be two points on a circle $\odot O$ with center O such that $\angle AOB = 154°$. D is a point outside $\odot O$ such that DA is tangent to $\odot O$ and $\angle DAB < 90°$. Find $\angle DAB$, answer in degrees.

Solution: 77°. Pick any point C on the major arc $\overset{\frown}{AB}$. Then $\angle ACB = \frac{1}{2}\angle AOB = 77°$, and $\angle DAB = \angle ACB$.

4. Starting with a given positive integer n, add all of its digits to obtain a new integer, and continue this procedure until we obtain a single digit number $f(n)$. For example, given $n = 1234$, we first get $1 + 2 + 3 + 4 = 10$, then $1 + 0 = 1$, so $f(n) = 1$. How many of the following 9 statements are correct?

- If $f(n) = 1$, then n must be a multiple of 1.
- If $f(n) = 2$, then n must be a multiple of 2.

$\vdots$

- If $f(n) = 9$, then n must be a multiple of 9.

> **Solution:** 3. The statement for 1 is obviously correct and the statements for 3 and 9 are true from the well-known criteria for divisibility by 3 and 9; while $11, 13, 14, 15, 16, 17$ serve the counterexamples for the remaining statements, respectively.

5. Tom can paint a fence in 15 hours by himself. Ben can paint the same fence in 10 hours by himself. Tom started painting the fence alone, but then tricked Ben into painting while Tom left. In total it took 11 hours to paint the fence. How many hours did Tom work?

> **Solution:** 3. Let x be the time Tom worked. Then $\frac{1}{15}x + \frac{1}{10}(11 - x) = 1$, solving gives $x = 3$.

6. Evaluate $\frac{1}{200}(51^2 + 52^2 + \ldots + 99^2 - 1^2 - 2^2 - \ldots - 49^2)$.

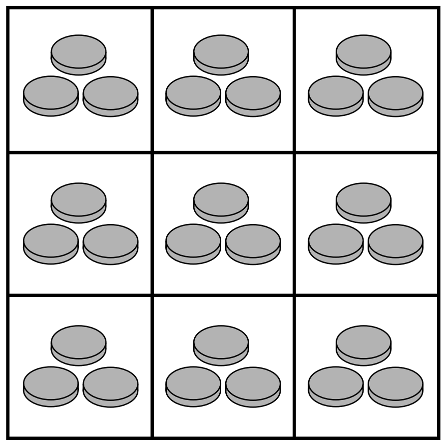
> **Solution:** 1225. Note that for each $i = 1, 2, \ldots, 49$, $\frac{1}{200}((100 - i)^2 - i^2) = 50 - i$, so the sum equals $1 + 2 + \ldots + 49 = \frac{50 \times 49}{2} = 1225$.

7. In a math competition a problem reads, "Find the smallest positive integer N such that, when N is divided by $7, 9$, or 32, the remainders are $6, 1$, or 8, respectively." Langston incorrectly computed $N = 1126$, although his only mistake was to read the last number as 6 instead of 8. What is the answer to the original question?

> **Solution:** 1000. Say the answer to the original question is $1126 - M$, then M shall be an integer which is a multiple of both 7 and 9, and $M + 2$ is a multiple of 32. Note that $7 \times 9 = 63$ itself satisfies that $63 + 1$ is a multiple of 32, so $2 \times 63 = 126$ is a valid M. As $7 \times 9 \times 32 = 2016 > 1000$, 1000 is indeed the smallest positive solution.

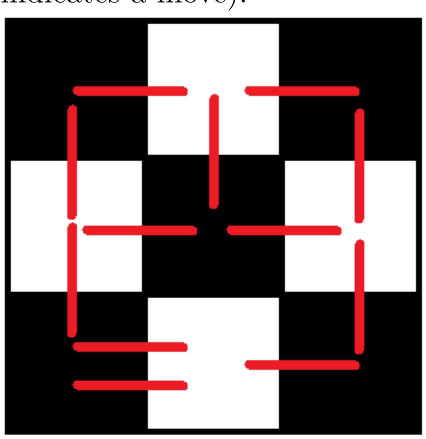
8. Choose four points A, B, C, and D on a circle uniformly at random. What is the probability that the lines AB and CD intersect outside the circle? Give your answer as the closest integer to (probability $\times$ 1000).

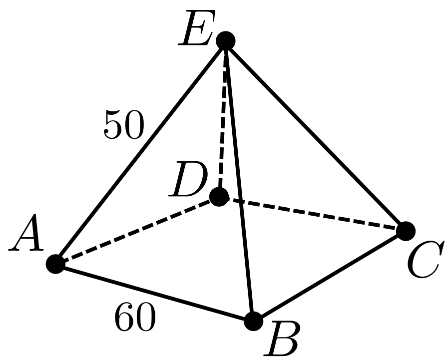
Solution: 667. Choose four unlabelled points on the circle and without loss of generality pick one of the points as A. Say the remaining points in clockwise order are X, Y, Z, and we have to assign labels B, C, D to them. The condition is satisfied if and only if we do not choose Y to be B, so the probability of an acceptable arrangement is $\frac{2}{3}$.



9. A square table is divided into a 3×3 grid with every cell having 3 coins. In every step of a game, Terry can take 2 coins from the table as long as they come from distinct but adjacent cells. (Here "adjacent" means the two cells share a common edge.) At most how many coins can Terry take?

Solution: 24. Every time Terry must take a coin from one of the four cells adjacent to the center, so he can take coins for at most $4 \times 3 = 12$ steps, hence at most 24 coins. The following scheme shows taking 24 coins away is possible (each red line indicates a move):





10. Consider a pyramid whose faces consist of a 60×60 square base $ABCD$ and four $60 - 50 - 50$ triangles that join at the apex E. If you are only allowed to move on the surfaces of the four triangles, what is the length of the shortest path between A and C?

> **Solution:** 96. Unfold the pyramid (without the base) and the shortest $A - C$ path is a straight line between them, say the line passes through BE, then the line must be perpendicular to it. The area of each triangle is 1200, so the length of the perpendicular from A to BE is $2 \times \frac{1200}{BE} = 48$. The distance from C to the same point is identical, in total $2 \times 48 = 96$.

11. The World Series is a best of seven competition that ends when one team has won four games. The Red Sox and the Braves each team have a 50% chance of winning any game they play. So far the Red Sox have won two games and the Braves have won one game. What is the chance that all seven games of the world series will be played? Give your answer as the closest integer to (probability $\times$ 1000).

> **Solution:** 375. If the Red Sox win on the seventh game, then the Braves must win exactly two of the three preceeding games and there are $\binom{3}{2}$ ways for this to happen. Similarily, if the Braves win on the seventh game, then the Braves must also win exactly win two of the three preceeding games, contributing 3 ways here. Each way happens with a probability of $\left(\frac{1}{2}\right)^4$. Therefore, the probability that all seven games will be played is $\frac{3}{8}$.

12. Let $N = 10^{10^{10}}$. Suppose $N^4 = 10^{10^{10+s}}$. Find $(\sqrt{10})^s$.

> **Solution:** 2. $N^4 = 10^{4 \times 10^{10}} = 10^{10^{(10+\log_{10} 4)}}$. So $s = \log_{10} 4 = 2 \log_{10} 2$, and $(\sqrt{10})^s = 10^{\frac{1}{2} \cdot 2 \log_{10} 2} = 10^{\log_{10} 2} = 2$.

13. It is known that $99999 = 41 \times 2439$. Suppose you take the $1^{st}, 2^{nd}, 4^{th}, 8^{th}, \ldots$ decimal digit of $\frac{1}{41}$ and list them in the original order to form a new decimal number $0.02\ldots$. Write the resulting number as a fraction $\frac{m}{n}$ in the lowest terms. What is $m + n$?

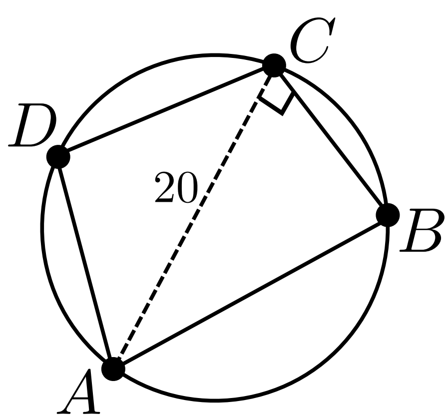
> **Solution:** 1137. We have $\frac{1}{41} = \frac{271\times 9}{99999}$, so $\frac{1}{41} = 0.\overline{02439}$. Since $2^n \pmod 5$ is periodic as $1, 2, 4, 3, 1, \ldots$, the new decimal number is $0.\overline{0234} = \frac{234}{9999} = \frac{26}{1111}$.

14. I have 5 (identical) apples and 9 (identical) bananas. How many ways are there for me to hand out all 14 fruits to 4 children, so that no child has more apples than bananas?

> **Solution:** 1960. First hand out the apples. This is the famous "stars and bars" problem with $\binom{5+4-1}{4-1}$ ways. Next hand out 5 bananas so that each child has as many bananas as apples. Then to hand out the remaining 4 bananas is a similar problem with $\binom{4+4-1}{4-1}$ ways. Hence the answer is $\binom{8}{3}\binom{7}{3}$.

15. A P-sequence is a sequence $a_1 = 1, a_2, \ldots, a_r = 2016$ such that each $\frac{a_{i+1}}{a_i}$ is a prime number. Find the minimum possible value of $a_1 + a_2 + \ldots + a_r$ among all P-sequences.

> **Solution:** 2463. We should assume $\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_r}{a_{r-1}}$ are non-decreasing: suppose not, say $p = \frac{a_{i+1}}{a_i} < \frac{a_i}{a_{i-1}} = q$, then $a_1 = 1, a_2, \ldots, a_{i-1}, pa_{i-1}, a_{i+1}, \ldots, a_r = 2016$ is a P-sequence with a smaller weight than $a_1 = 1, a_2, \ldots, a_{i-1}, a_i = qa_{i-1}, a_{i+1}, \ldots, a_r = 2016$. Since $2016 = 2^5 \times 3^2 \times 7$, the P-sequence with the minimum weight is 1, 2, 4, 8, 16, 32, 96, 288, 2016 with weight 2463.



16. Let $ABCD$ be a trapezoid with AB parallel to CD and $AC = BD = 20$ and $\angle ACB = 90°$. Suppose the trapezoid can be inscribed in a circle with area 125π. What is the area of this trapezoid?

Solution: 160. $\angle ACB = 90°$ implies AB is the diameter of the circle with length $2\sqrt{\frac{125\pi}{\pi}} = 10\sqrt{5}$, so $CB = \sqrt{AB^2 - AC^2} = 10$. Let E and F be the points on AB such that $CE, DF \perp AB$. The fact that $\triangle ACB \sim \triangle CEB$ gives $CE = 4\sqrt{5}$ and $BE = 2\sqrt{5}$. The trapezoid is symmetric as it is inscribed in a circle, thus $AF = BE$. Finally $CD = EF = AB - AF - BE = 6\sqrt{5}$, and the area is $\frac{(CD+AB)CE}{2} = 160$.

17. How many integers between 0 and 100 (inclusive) can be written in the form $\lfloor 2x \rfloor \cdot \lfloor 3x \rfloor$ for some number $x \geq 0$? (Here $\lfloor y \rfloor$ is the greastest integer no larger than y.)

Solution: 16. Consider the interval partition $[0, \frac{1}{3}), [\frac{1}{3}, \frac{1}{2}), [\frac{1}{2}, \frac{2}{3}), [\frac{2}{3}, 1), [1, \frac{4}{3}), [\frac{4}{3}, \frac{3}{2}), \ldots$. The value of $\lfloor 2x \rfloor \cdot \lfloor 3x \rfloor$ is monotonic increasing and exceeds 100 for $x \geq \frac{13}{3}$. Thus it suffices to consider the first 17 intervals. Except for the transition between the first two intervals (in which $\lfloor 2x \rfloor \cdot \lfloor 3x \rfloor$ are both 0), the function $\lfloor 2x \rfloor \cdot \lfloor 3x \rfloor$ strictly jumps at other transitions because either $\lfloor 2x \rfloor$ or $\lfloor 3x \rfloor$ would increase by 1 while the other term is positive, so we have 16 distinct values.

18. Jane rolls a fair, standard six-sided die repeatedly until she rolls a 1. She begins with a score of 1, and each time she rolls x, her score is divided by x. Let $\frac{m}{n}$ be the expected value of Jane's final score as a fraction in the lowest terms. What is $m + n$?

Solution: 111. Let X be the product of the reciprocal of the rolls. Then $E[X]$ can be related to itself by considering the result of the first roll as follows. Either Jane rolls a 1 and the game stops, or with probability $\frac{1}{6}$ each she rolls n for $n = 2, 3, 4, 5, 6$ and the game essentially starts again with a new initial score. So $E[X] = \frac{1}{6}(1) + \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right)E[X]$, solving gives $E[X] = \frac{20}{91}$.

19. Let $R(x)$ be the remainder when the polynomial $P(x) = \sum_{n=1}^{2016} x^{2n^2}$ is divided by $x^3 - x$. Find $R(2)$.

Solution: 8064. Let $Q(x)$ be the quotient when $P(x)$ is divided by $x^3 - x$. Then $(x^3 - x)Q(x) + R(x) = P(x)$ for all x. Plugging in $x = -1, 0, 1$ gives $R(-1) = P(-1) = 2016$, $R(0) = P(0) = 0$, and $R(1) = P(1) = 2016$. Since $x^3 - x$ has degree 3, $R(x)$ has degree at most 2. The only quadratic (or lower degree) polynomial which satisfies $R(-1) = 2016$, $R(0) = 0$, and $R(1) = 2016$ is $R(x) = 2016x^2$. Thus, $R(2) = 2016 \cdot 2^2 = 8064$.

20. Let z be a complex number such that $|z-156|+|z+65i|=169$. Find the minimum possible value of $|z|$. (Here $|z|$ is the absolute value of z.)

> **Solution:** 60. By the triangle inequality, $|z-156|+|z+65i| \geq |(z-156)-(z-65i)| = |-156+65i| = 169$ with equality holds if and only if z lies on the line segment between the points $A = 156$ and $B = 65i$. Hence we want find the minimum distance from the origin $O = 0$ to the line segment AB, that is, the length of the altitude of $\triangle OAB$ from O to AB. Since $\triangle OAB$ is a right triangle, its area S_{OAB} equals $\frac{1}{2}|OA||OB| = \frac{1}{2} \cdot 156 \cdot 65 = 5070$; as $|AB| = 169$, the length of the altitude is $\frac{2S_{OAB}}{|AB|} = 60$.

21. Find the maximum possible value for $m+n$ with m, n being positive integers, subject to $m+n < 10000$ and $\mathrm{lcm(m,n)} - 18\,\mathrm{gcd(m,n)} = \mathrm{m+2n}$.

> **Solution:** 9996. Divide both sides by $\gcd(m,n)$, we have $\frac{m}{\gcd(m,n)}\frac{n}{\gcd(m,n)} - 18 = \frac{m}{\gcd(m,n)} + 2\frac{n}{\gcd(m,n)}$. Denote by $m' := \frac{m}{\gcd(m,n)}, n' := \frac{n}{\gcd(m,n)}$, then $(m'-2)(n'-1) = 20$ with $\gcd(m', n') = 1$. Among all pairs (a, b) with $ab = 20$, only $(5, 4), (2, 10)$ satisfy $\gcd(a+2, b+1) = 1$. Hence (m, n) is a solution to the original equation if and only if it is of the form $((a+2)d, (b+1)d)$ for some such (a, b) and some d, and $m+n = (a+b+3)d$. The largest multiples of $5+4+3 = 12, 2+10+3 = 15$ that are less than 10000 are $9996, 9990$, respectively.

22. Suppose that $f(x)$ is a positive real function defined on all real numbers such that (1) the second derivative $f''(x)$ of $f(x)$ exists and is continuous, and (2) $f(x)$ achieves its maximum value of 1 at every positive divisor of 2016. What is the minimal number of roots to the equation $f''(x) = 0$?

> **Solution:** 72. Since $2016 = 2^5 \times 3^2 \times 7$, 2016 has 36 divisors, i.e. $f(x)$ has at least 36 maxima with $f'(x) = 0$ there. Moreover, as $f(x) = 1$ at those divisors, $f'(x) = 0$ must also have at least a root between every consecutive pair of divisors by Rolle's theorem. So $f'(x) = 0$ has at least 71 roots.
>
> By Rolle's theorem, $f''(x) = 0$ has a root between every pair of consecutive roots of $f'(x) = 0$, giving at least 70 roots of $f''(x) = 0$ on the interval $(1, 2016)$. Furthermore, since $f''(1), f''(2016) \leq 0$ while $f'(1) = f'(2016) = 0$, there must also be points with non-negative concavity above 2016 and below 1, otherwise $f(x)$ could not remain positive. They correspond to a root of $f''(x) = 0$ on each of $(-\infty, 1]$ and $[2016, \infty)$.
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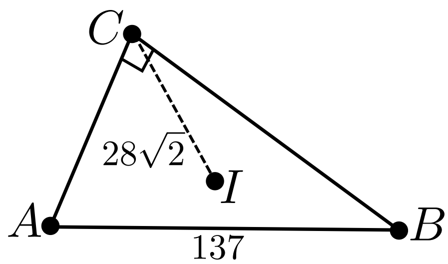
> The above show $f''(x) = 0$ has at least 72 roots. Such a lower bound can be achieved by an appropriately scaled sum of Gaussians centered at the divisors.



23. Suppose $\triangle ABC$ is a right triangle with $\angle C = 90°$, $AB = 137$, and $CI = 28\sqrt{2}$, where I is the incenter of $\triangle ABC$. Find the area of the triangle. (The incenter is the intersection of the angle bisectors.)

Solution: 4620. Write $AB = c, BC = a, CA = b$, and say the incircle of $\triangle ABC$ tangent to AB, BC, CA at R, P, Q, respectively. Then $IPCQ$ is a sqaure as all angles are right angles and $IP = IQ$, so $CI = \sqrt{2}PC = \frac{a+b-c}{\sqrt{2}}$ thus $a + b = \sqrt{2}CI + c = 193$, here we could compute PC by noting $AQ = AR, BR = BP, CP = CQ$ and $BP + CP = a, AQ + PQ = b, AR + RB = c$. Now the area of $\triangle ABC$ equals $\frac{ab}{2} = \frac{(a+b)^2-(a^2+b^2)}{4} = \frac{193^2-137^2}{4} = \frac{(193+137)(193-137)}{4} = 4620$.

24. Everyone is either a knight or a knave. Every statement a knight says must be true, while every statement a knave says is false. There are 12 people: $A_1, \ldots, A_{12}$. For each $i = 3, \ldots, 12$, the first knight A_1 makes the following statement: "A_i is a knight and says that A_{i-1} says that A_{i-2} says that $\ldots A_2$ says that A_1 is a knave." How many possible compositions of knights and knaves are there for these 12 people?

Solution: 234. Write 0 for knight and 1 for knave. Suppose that $A_1 = 0$. Then A_1's statements are correct, thus for all $i = 2, \ldots, 12$ we have $\sum_{j=1}^{i} A_j = 1 \pmod 2$. Therefore $A_2 = 1, A_3 = \ldots = A_{12} = 0$ is the only possibility.
Now suppose $A_1 = 1$. Then A_1's false statements say that $A_i = 0 \pmod 2$ and $\sum_{j=1}^{i-1} A_j = 1 \pmod 2$. The truth must be that, for all $i = 3, \ldots, 12$, $\sum_{j=1}^{i-1} A_j = 0 \pmod 2$ or $\sum_{j=1}^{i} A_j = 0 \pmod 2$. So the possibilities biject to length 11 bit strings $A_2, \ldots, A_{12}$ whose partial sums $A_2, A_2 + A_3, \ldots, A_2 + \ldots + A_{12} \pmod 2$ do not have consecutive zeros, or equivalently to bit strings without two consecutive 1's. It is well-known that the answer to this problem is the 12-th Fibonacci number (we use the convention $F_0 = F_1 = 1, F_n = F_{n-1} + F_{n-2}$). Summing up, the total number of possibilities is $F_{12} + 1 = 234$.

25. An airline operates flights between some cities. We say two cities A and B are "connected" if there are direct flights between them. Moreover, the following are true:

   1. Each city is connected to exactly 16 other cities.

2. For any two connected cities C_1, C_2, there are exactly 8 other cities connected to both C_1 and C_2.
3. For any two cities D_1 and D_2 that are not connected, there are exactly 4 other cities connected to both D_1 and D_2.

A 3-group is three distinct, pairwise connected cities C_1, C_2, C_3 (we consider different orderings of the same three cities the same 3-group). How many 3-groups are there?

Solution: 960. We first find the number of cities, denoted by n. Fix a city A, we count the number of pairs (C, D) where A, C and C, D are connected but not A, D. On the one hand there are $n - 16 - 1$ cities that are neither A itself nor connected to A, each of them is connected to 4 cities that are connected to A, so we have $4(n-17)$ such pairs. On the other hand, there are 16 cities connected to A and each of them has direct flights to $16 - 1 - 8 = 7$ cities that are not connected to A, so $4(n - 17) = 16 \times 7$, solving gives $n = 45$.

Now we count the number of 3-group. Including order, there are 45 choices for C_1, 16 choices for C_2 and 8 choices for C_3. But we do not care ordering of the cities, so the answer is $\frac{45\times16\times8}{3!} = 960$.

Remark: Such flight arrangement exists, say represent each city by two distinct numbers from $1, 2, \ldots, 10$, then add direct flights between every pair of cities that share a number in their representations.