Georgia Tech High School Math Competition

Ciphering Test—Solutions

February 28, 2015

1. How many digits does 2015 have in base 5?

> **Solution:** $5^5 > 2015 > 5^4$ so it has 5 digits.

2. Consider a convex N-gon whose interior angles are integers when measured in degrees. What is the maximum possible value of N?

> **Solution:** Each angle can be at most 179°. Such a case can occur when the polygon is a regular 360-gon. So the answer is 360.

3. The 20 digit decimal number $60028022015X28022015$ is divisible by 11. What is the missing digit X?

> **Solution:** As $10 \equiv -1 \pmod{11}$, the digits must alternate sum to a number divisible by 11. So $X \equiv 6 \pmod{11}$, and $X = 6$.

4. Two 6-sided dice and one 8-sided die are placed in a bag. A die is selected at random and rolled. The result is a 4. What is the probability that the rolled die had 6 sides? *Express your answer as a reduced fraction.*

> **Solution:** The probability of rolling a 4 is $p_1 = \frac{2}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{8} = \frac{11}{72}$. The probability of selecting a 6 sided die and rolling a 4 is $p_2 = \frac{2}{3} \cdot \frac{1}{6} = \frac{1}{9}$. Hence, the probability that the rolled die was six sided is $\frac{p_2}{p_1} = \frac{8}{11}$.

5. Inside the unit circle C_1 is inscribed an equilateral triangle T_1, inside of which is inscribed a circle C_2 inside of which is inscribed an equilateral triangle T_2, inside of which is inscribed a circle C_3, ... What is the side length of triangle T_6?

> **Solution:** At each stage the figure is scaled by a factor of $\frac{1}{2}$ and the first triangle has side length $\sqrt{3}$. Thus the side length of triangle T_6 is $\frac{\sqrt{3}}{32}$.

6. It is known that in the sequence $A, X, B, C, D, Y, 11$ the sum of any 3 consecutive terms is 19. Find the value of $A + B + C + D$.

> **Solution:** 30. By assumption $B+C+D = 19$. Now since $A+X+B = X+B+C = 19$ we can deduce that $A = C$. Analogously since $C + D + Y = D + Y + 11 = 19$ we can deduce that $C = 11$. Combining the two we get $A = C = 11$ and so the answer is $A + B + C + D = 19 + 11 = 30$.

7. A square is inscribed in a circle with integer radius. Let A be the sum of the areas of the square and the circle and let B be the sum of the circumference and the diameter of the circle. If $A - B$ is an integer, what is its value?

> **Solution:** Let r be the length of the radius. We know that r and $A - B = \pi(r^2 - 2r) + 2r^2 - 2r$ are integers. As π is irrational r must be equal to 2 and the value sought is 4.

8. What is the smallest positive integer with exactly 12 positive divisors?

> **Solution:** A number $N = p_1^{a_1} \dots p_d^{a_d}$ has exactly $(a_1+1)\dots(a_d+1)$ positive divisors. Note that the number of positive divisors does not depend on the primes in the prime decomposition but only on their powers. Thus to find the smallest positive integer with a given number of positive divisors we may assume that $a_1 \geq a_2 \geq \dots \geq a_d$ and that p_i is the i^{th} smallest prime number. Now the factorizations of 12 are $12 \times 1, 6 \times 2, 4 \times 3$, and $3 \times 2 \times 2$. The positive integers corresponding to each factorization, given our assumptions above, are $2^{11} = 2048, 2^5 \times 3 = 96, 2^3 \times 3^2 = 72, 2^2 \times 3 \times 5 = 60$. Comparing these integers we see that the smallest positive integer with exactly 12 positive divisors is 60.

9. A round table with 5 place settings is to be decorated. Each place setting may be decorated in one of 5 ways. How many distinct table settings are there?

> **Solution:** As there are 5 options for each place setting there are 5^5 table settings before we account for the rotational symmetry. There are 5 table settings with each place setting the same. As 5 is prime these are exactly the table settings invariant under rotation. Hence, there are $\frac{5^5 - 5}{5} + 5 = 629$ distinct settings.

10. Let $a_1, a_2, \ldots, a_{10}$ be positive integers whose sum is 30. What is the maximum possible value of $a_1a_2 + a_2a_3 + \cdots + a_9a_{10}$?

> **Solution:** Let $M = a_1a_2 + \cdots + a_9a_{10}$. Observe that for each $1 \leq i \leq 7$ we have $a_ia_{i+1} + a_{i+1}a_{i+2} = (a_i - 1)a_{i+1} + a_{i+1}(a_{i+2} + 1)$. So if we replace a_i and a_{i+2} by $(a_i - 1)$ and $(a_{i+2} + 1)$, respectively, M increases by $a_{i+3} - 1 \geq 0$ except for the case $i = 1$ when it remains the same. Furthermore the constraint $a_1 + \cdots + a_{10} = 30$ still holds. Thus it suffices to consider the case $a_1 = a_2 = \cdots a_7 = 1$. Now we must maximize the sum $6 + a_8 + a_8a_9 + a_9a_{10}$. We observe that by replacing a_8 and a_{10} by $a_8 + 1$ and $a_{10} - 1$, respectively, M increases by 1 and the constraint $a_1 + \cdots + a_{10} = 30$ still holds. Thus we just have to maximize $6 + a_8 + a_8a_9 + a_9$ where $a_8 + a_9 = 22$. This is equivalent to maximizing the quadratic $29 + a_8(22 - a_8)$. The maximum value of this function is 149 and is achieved when $a_8 = 11$.