2019 Georgia Tech High School Mathematics Competition
Proof Test

Instructions. Congratulations on advancing to the proof portion of the competition! Do not open this envelope until instructed to do so. Please write your 5-digit ID number in the designated area at the bottom of this envelope.

Contestant ID Number:
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1. Given that

$$\log_2(11) = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{25 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}}}}}}$$

Determine, with proof, which of 2^{128} and 11^{37} is larger.

Solution: 2^{128} is larger. The goal is to show that $37\log_2(11) < 128$ using the continued fraction. To that end, note that

$$\frac{128}{37} = 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{2}}}} > 3 + \cfrac{1}{2 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \text{a positive quantity}}}}}} = \log_2(11)$$

Note that adding a positive quantity to the n-th denominator makes a continued fraction alternatively bigger or smaller depending on the parity of n. I.e.

$$a < a + \frac{1}{\text{pos.}}, \qquad a + \frac{1}{b} > a + \cfrac{1}{b + \cfrac{1}{\text{pos.}}}, \qquad a + \cfrac{1}{b + \cfrac{1}{c}} < a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{\text{pos.}}}}$$

and so on. (It's smaller with an even number of horizontal lines on the left and bigger with an odd number.)

2. A spanning tree in a graph is a collection of edges that doesn't create any loops and such that it is possible to walk along the edges from one vertex in the graph to any other vertex in the graph.

   Derive (with proof) a formula for the number of spanning trees in the graph $K_{2,n}$ pictured in Figure 1

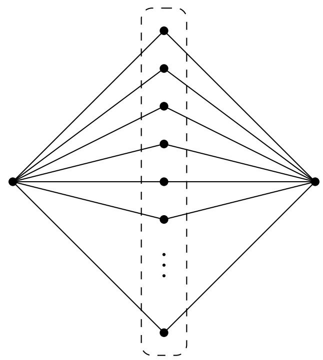

Figure 1: $K_{2,n}$

> **Solution:** There are $n2^{n-1}$ spanning trees. Each vertex in the middle needs to have an edge. If two or more middle vertices have both the left and right edges, then we create a loop. If no middle vertices have both a left and right edge, then it is impossible to get from the left vertex to the right vertex.
>
> So there is exactly one middle vertex having both the left and right edge in the tree. There are n choices for this.
>
> This leaves $n-1$ remaining middle vertices and there are two choices (left or right) for each of them.

3. How many functions $f : \mathbb{Z} \to \mathbb{Z}\backslash\{0\}$ satisfy the following:

$$f(m+n) = f(m)/f(n) \qquad \text{for all } m, n \in \mathbb{Z}$$

Solution: Consider $m = n = 0$. Then,

$$f(0) = f(0)/f(0) = 1$$

and thus we have that $f(0) = 1$. Letting $m = 0$, we see for all n that

$$\begin{aligned} f(n) &= f(0)/f(n) = 1/f(n) \\ \Rightarrow f(n)^2 &= 1 \\ \Rightarrow f(n) &= \pm 1 \end{aligned}$$

that is that $f(n)$ is either 1 or -1 for all n. Letting $m = n$ we have that for all m:

$$\begin{aligned} f(m+m) &= f(m)/f(m) \\ \Rightarrow f(2m) &= 1, \end{aligned}$$

that is to say that $f(n) = 1$ for all even n. Letting $n = 2$ and using that $f(2) = 1$ from the previous observation, we see that for all m:

$$f(m+2) = f(m)/f(2) = f(m)/1 = f(m).$$

It isn't difficult to see that the previous observation implies that all the odd numbers have the same function value (in particular, they all have the function value of 1 or they all have the function value of -1). Since we know that $f(n) = 1$ for all even n this gives us two possibilities for what f can be:

$$f(n) = 1$$

or

$$f(n) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases} = (-1)^n.$$

It is easily verified that both of these functions satisfy the functional equation.

4. Let E be a finite set of points in the plane that don't all lie on the same line. Let L be the set of lines formed by two or more of the points. Prove that $\#E \leq \#L$ and describe (without proof) which arrangements have $\#E = \#L$.

   You may use the following theorem.

   Theorem: [Tibor Gallai, 1944] Given any finite set E of points in the plane, that don't all lie on the same line. There exists a pair of points P, Q such that the line $\overline{PQ}$ contains only P and Q.

   Solution: The minimal case is three points, not all in a line in which we have 3 lines as well.

   Now assume that the statement is proved for $n-1$ points and we will show it for n-points.

   Let $\overline{P_1P_2}$ be a line containing only P_1 and P_2. Let $P_3, \ldots, P_n$ be the other points in E.

   Consider the subset $\{P_2, \ldots, P_n\}$. If these are not all on a line, then by induction, they determine at least $n-1$ lines. So together with $\overline{P_1P_2}$ (which is not a line formed by any pair of $P_2, \ldots, P_n$), we have at least n lines.

5. The Fibonacci sequence is defined by $F_0 = F_1 = 1$ and $F_{k+1} = F_k + F_{k-1}$ for all positive integers k. Define a *golden polynomial* to be any polynomial in the form $\sum_{k=0}^{n} \epsilon_k F_k x^{n-k}$ where each ϵ_k is either $+1$ or -1. For example, the polynomials $x-1$, $-x^2+x+2$, $-x^4-x^3+2x^2-3x+5$ and $x^6+x^5-2x^4+3x^3+5x^2-8x+13$ are all golden polynomials. Find with proof the largest possible degree of a golden polynomial that has only real roots.

   Solution: The degree 2 golden polynomial $x^2-x-2 = (x-2)(x+1)$ has only real roots. Now, suppose that there is a golden polynomial $\epsilon_0 x^n + \epsilon_1 x^{n-1} + 2\epsilon_2 x^{n-2} + \cdots + \epsilon_n F_n$ of degree $n \geq 3$. Without loss of generality, we can assume the leading coefficient is $\epsilon_0 = +1$ since flipping all the signs doesn't change the roots. Using Vieta's relations, the real roots $r_1, \ldots, r_n$ satisfy $\sum_{k=1}^{n} r_k = -\epsilon_1$, $\sum_{1 \leq k < \ell \leq n} r_k r_\ell = 2\epsilon_2$, and $\prod_{k=1}^{n} r_k = (-1)^n \epsilon_n F_n$. Hence, the sum of the squares of the roots is $\sum_{k=1}^{n} r_k^2 = \left(\sum_{k=1}^{n} r_k\right)^2 - 2\left(\sum_{1\le k<\ell\le n} r_k r_\ell\right) = (-\epsilon_1)^2 - 2(2\epsilon_2) = 1 - 4\epsilon_2 = \begin{cases} 5 & \text{if } \epsilon_2 = -1 \\ -3 & \text{if } \epsilon_2 = +1 \end{cases}$. Since the roots are real, the sum of the squares of the roots must be nonnegative. Hence, $\epsilon_2 = -1$, and $\sum_{k=1}^{n} r_k^2 = 5$. But then, using the RMS-GM inequality, we have: $\sqrt{\frac{5}{n}} = \left(\frac{1}{n}\sum_{k=1}^{n}|r_k|^2\right)^{1/2} \geq \left(\prod_{k=1}^{n}|r_n|\right)^{1/n} = F_n^{1/n}$, and thus, n must satisfy $nF_n^{2/n} \leq 5$. However, for $n \geq 5$, we have $nF_n^{2/n} > n \geq 5$. Also, for $n = 4$ and $n = 3$, we have $4F_4^{2/4} = 4 \cdot 5^{1/2} > 5$ and $3F_3^{2/3} = 3 \cdot 3^{2/3} = 3^{5/3} > 5$ (since $3^5 > 5^3$). Therefore, there are no golden polynomials with degree $n \geq 3$ that only have real roots. Hence, the largest possible degree of a golden polynomial with only real roots is 2.

6. (Tiebreaker) For any prime number p, is it true that there is a Fibonacci number F_n with $n \geq 1$ which is divisible by p? Prove or disprove.

Solution: Note that the Fibonacci sequence is periodic mod p.

First note that there are only finitely many pairs (a, b) of integers mod p. So there must be at least two different places in the Fibonacci sequence where a and b are consecutive. But a pair consecutive Fibonacci numbers determines the entire sequence (both forwards and backwards) since given F_i, F_{i+1} we have

$$\begin{aligned} F_{i+2} &\equiv F_i + F_{i+1} \pmod{p} \\ F_{i-1} &\equiv F_{i+1} - F_i \pmod{p}. \end{aligned}$$

Thus if we have $(F_i, F_{i+1}) \equiv (a, b) \pmod{p}$ and $(F_{i+N}, F_{i+N+1}) \equiv (a, b) \pmod{p}$, then also $(F_{i+2N}, F_{i+2N+1}) \equiv (a, b) \pmod{p}$ and $(F_{i+3N}, F_{i+3N+1}) \equiv$

$(a, b) \pmod p$ and so on, for any integer multiple of N. Here we allow our Fibonacci numbers to be indexed by negative numbers as well.

Now given that the Fibonacci sequence is periodic mod p and $F_0 = 0$, it must be that $F_N \equiv 0 \pmod p$ if the period is N.

End of exam.

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