Georgia Tech High School Math Competition

Multiple Choice Test

April 13, 2019

- Each correct answer is worth one point; there is no deduction for incorrect answers.
- Make sure to enter your ID number on the answer sheet.
- You may use the test booklet as scratch paper, but no credit will be given for work in the booklet.
- You may keep the test booklet after the test has ended.

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1. Let f be some unknown exponential function (i.e. it has the form $c \cdot a^x$ for some real constants $c \neq 0$ and $a > 0$). If $f(3) = 2$ and $f(5) = 18$, what is $f(4)$?
   (a) 4
   (b) 6
   (c) 8
   (d) 10
   (e) 12

Solution: B: 6. Exponential functions obey the recurrence:

$$f(x+1) = af(x)$$

if $f(x) = c \cdot a^x$. Thus,

$$18 = f(5) = af(4) = a^2 f(3) = 2a^2$$

which implies that $a = 3$. Thus,

$$f(4) = af(3) = 3 \cdot 2 = 6.$$

2. How many integers between 1 and 999 are divisible by 3 or 5 but not by 3^3?
   (a) 428
   (b) 429
   (c) 465
   (d) 466
   (e) 561

Solution: B: 429 There are 333 numbers divisible by 3, 199 which are divisible by 5, 66 by $3 \cdot 5$, and 37 by 3^3.

So by Inclusion-Exclusion, there are

$$(333 + 199 - 66) - 37 = 429$$

such numbers.

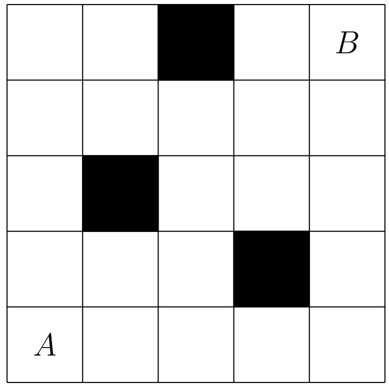


Figure 1: Problem 3

3. How many paths are there in Figure 1 from square A to square B consisting of only up and right steps which don't visit any of the black squares.

   (a) 11
   (b) 14
   (c) 17
   (d) 18
   (e) 21

Solution: D: We can solve this problem recursively by counting the number of paths which start at A and end at each square without visiting a black square. Obviously, there is one valid path from A to itself, and zero valid paths from A to any black square. For any other square, the number of valid paths is the number of valid paths for the square immediately to its left plus the number of valid paths for the square immediately below. The following table shows the number of valid paths that reach each square. The number of valid paths to B is $\boxed{18}$.

1	1		7	18
1	1	4	7	11
1		3	3	4
1	2	3		1
1	1	1	1	1

4. Which of the following numbers has a finite decimal expansion?

$$\frac{2019}{120}, \quad \frac{2019}{350}, \quad \frac{2019}{603}, \quad \frac{2019}{900}, \quad \frac{2019}{9009}.$$

   (a) $\frac{2019}{120}$
   (b) $\frac{2019}{350}$
   (c) $\frac{2019}{603}$
   (d) $\frac{2019}{900}$
   (e) $\frac{2019}{9009}$

Solution: A: $\frac{2019}{120}$.

One can show that a reduced fraction has a finite decimal expansion if and only if the denominator is of the form $2^a \cdot 5^b$ for non-negative integers a and b. For the first fraction, observe that 3 divides both 120 and 2019 so the fraction reduces to $\frac{673}{40}$. This fraction can't be further reduced and the denominator is $40 = 2^3 \cdot 5^1$ so by the fact stated above, we have that the first fraction has a finite decimal expansion. A similar technique shows that the remaining fractions don't have a finite decimal expansion.

5. John rides his bike from his house to the store at 6 miles per hour. Upon arriving at the store, he realizes it's closed and immediately heads back, riding his bike back home at a speed of 2 miles per hour. What is John's average speed in mph?

   (a) 3
   (b) $\frac{7}{2}$
   (c) 4
   (d) $\frac{9}{2}$
   (e) 5

Solution: A: 3 miles per hour.

Let d be the distance from the house to the store. Let t_1 be the amount of time it took to go from the house to the store and let t_2 be the amount of time it took to go from the store to the house. Using the formula $d = vt$ where d is distance, v is a constant velocity, and t is time, we get the equations:

$$d = 6t_1$$
$$d = 2t_2$$

which together implies:

$$6t_1 = 2t_2 \implies t_2 = 3t_1.$$

Recall the equation for average speed is the ratio of the *total* distance and the *total* time. The total distance is $2d$ and the total time is $t_1 + t_2$. Using the equations we have above, we get that the average speed is:

$$\frac{2d}{t_1 + t_2} = \frac{2(6t_1)}{t_1 + (3t_1)} = \frac{12t_1}{4t_1} = 3 \text{ miles per hour.}$$

6. An ice cream store has n flavors. There are $d \cdot 100$ ways of choosing the flavors of two scoops of ice cream for some integer $d > 0$, where the order of flavors does not matter. What is the smallest possible value of n?
   (a) 21
   (b) 22
   (c) 23
   (d) 24
   (e) 25

Solution: D: 24 Suppose there are n flavors. There are n possible ways if we use the same flavor twice and $\binom{n}{2}$ ways if we use two different flavors (recall the order of the scoops don't matter). Thus, there are a total of $n + \binom{n}{2}$ ways. This expression is equal to the expression $\frac{n(n+1)}{2}$. In order for this quantity to be divisible by 100, the numerator must be divisible by $2 \cdot 100 = 200 = 2^3 \cdot 5^2$. It can't be the case that both n and $n+1$ are divisible by 5 since they are consecutive integers, so one of n or $n+1$ has to be divisible by 25 since the numerator has to be divisible by 25. It turns out if we let $n+1 = 25$, i.e., $n = 24$, then the expression $\frac{n(n+1)}{2} = 600$ which is divisible by 100. Additionally, we know we found the smallest possible value of n since $n = 24$ and we know that either n or $n+1$ must be a multiple of 25.

7. Let $f(x)$ be the polynomial function of smallest degree such that $f(0) = 2, f(1) = 1, f(2) = 12$ and $f(3) = 47$. Find $f(4)$.

(a) 78

(b) 92

(c) 118

(d) 156

(e) 228

Solution: C: $f(4) = 118$. This can be computed using finite differences.

$$\begin{array}{c|cccc}
0 & 2 & & & \\
 & & -1 & & \\
1 & 1 & & 12 & \\
 & & 11 & & 12 \\
2 & 12 & & 24 & \\
 & & 35 & & 12 \\
3 & 47 & & 36 & \\
 & & 71 & & \\
4 & 118 & & &
\end{array}$$

8. What is $1^2 + 2^2 + 3^2 + ... + 101^2 \mod 4$?
   (a) 0
   (b) 1
   (c) 2
   (d) 3

Solution: D: 3 The square of any even number is 0 mod 4, and the square of any odd number is 1 mod 4. Therefore, the above sum just counts the number of odd integers from 1 to 101, which is 51. $51 = 3 \mod 4$.

9. How many ways are there to connect the dots in Figure 2 with no lines overlapping and such that the big triangle is divided into triangles of area 1/2?

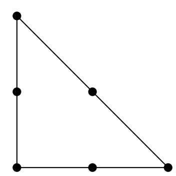
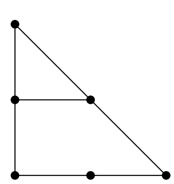


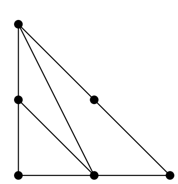
Figure 2: Triangle to be cut up.

   (a) 1
   (b) 2
   (c) 4
   (d) 6
   (e) 9

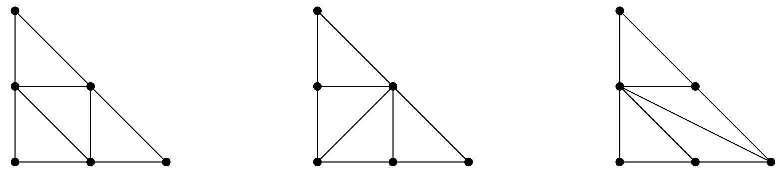
Solution: C: 4.
There are 4 ways to do this.

Starting from the top-most vertical edge, we have to form one of the following two triangles.





The first triangle can be further cut up in three ways and the second in just one.



10. Suppose b and c are chosen independently and uniformly at random from the set

$$\{-3, -2, -1, 0, 1, 2, 3\}.$$

What is the probability that the equation $x^2 + bx + c = 0$ has at least one real solution?

(a) $\frac{3}{7}$

(b) $\frac{27}{49}$

(c) $\frac{4}{7}$

(d) $\frac{34}{49}$

(e) 1

Solution: D: $\frac{34}{49}$ There are $7^2 = 49$ possible pairs of values which b and c can take. The quadratic equation $x^2 + bx + c = 0$ has at least one real solution if and only if the discriminant $b^2 - 4c$ is non-negative, i.e. $b^2 \geq 4c$.

If $c = -3$, -2, -1, or 0, then $b^2 \geq 0 \geq 4c$ for all 7 values of b.

If $c = 1$, then $b^2 \geq 4c = 4$ holds for 4 values of b (just $b = \pm 2$ or ± 3).

If $c = 2$, then $b^2 \geq 4c = 8$ holds for 2 values of b (just $b = \pm 3$).

If $c = 3$, then $b^2 \geq 4c = 12$ holds for 0 values of b.

Thus, there are $7 + 7 + 7 + 7 + 4 + 2 + 0 = 34$ pairs of values for b and c for which $x^2 + bx + c = 0$ has at least one real solution. Therefore, the probability that $x^2 + bx + c = 0$ has at least one real solution is $\boxed{\frac{34}{49}}$.

11. What is $2^{(2^{2019})} \pmod 7$?
    (a) 1
    (b) 2
    (c) 4
    (d) 5
    (e) 6

Solution: C: 4. By Fermat's Little Theorem, one can see that the answer is 2^c where $c = 2^{2019} \bmod 6$.

To calculate c, it suffices to find $y \in \{0, 1, 2, 3, 4, 5\}$ such that $2^{2019} \equiv y \pmod 6$. Observe that this only has a solution if y is even, that is $y = 2x$ for some integer x. Thus, we have the equation $2^{2019} \equiv 2x \pmod 6$. The solutions to this equation can be obtained by looking at the solutions of $2^{2018} \equiv x \pmod 3$. By Fermat's Little Theorem, since $2018 \bmod 2 = 0$, we have that $x \equiv 2^0 \equiv 1 \pmod 3$ and hence we have that $x \equiv 1, 4 \pmod 6$ and hence $y = 2x = 2(1) = 2 \pmod 6$ (one would get the same result for y if we plugged in $x \equiv 4$.) Thus, we have that $c = 2$.

Thus, the answer is $2^c = 2^2 = 4$.

12. Consider a triangle. It has three vertices v_1, v_2, v_3 and a center C. You start at vertex v_1 and you wish to go to vertex v_3. At every time step, you uniformly at random move to one of the four points (e.g. v_1, v_2, v_3, C) that you currently aren't at. In addition, you can't move to the center point if you were there the previous time step (i.e. you can't go from point C to vertex v_i and then back to point C for any $i = 1, 2, 3$). Calculate the expected number of time steps it will take to go from vertex v_1 to vertex v_3.
    (a) 1.8
    (b) 2
    (c) 2.2
    (d) 2.5
    (e) 2.8

Solution: E: 2.8

For any point p, let $E(p)$ denote the expected number of time steps it'll take to move from p to v_3. We wish to find $E(v_1)$. Observe that $E(v_3) = 0$. Let $E(p')$ denote the expected number of time steps it'll take to move from p to v_3 given that we're not allowed to move to C in the first step. By applying a recurrence relation, we get the following:

$$E(v_1) = \frac{E(v_2)+1}{3} + \frac{E(C)+1}{3} + \frac{E(v_3)+1}{3} = 1 + \frac{1}{3}E(v_2) + \frac{1}{3}E(C) + \frac{1}{3}E(v_3).$$

Similarly, we find:

$$E(C) = 1 + \frac{1}{3}E(v_1') + \frac{1}{3}E(v_2') + \frac{1}{3}E(v_3)$$

and

$$E(v_1') = 1 + \frac{1}{2}E(v_2) + \frac{1}{2}E(v_3).$$

By a symmetry argument, $E(v_1) = E(v_2)$ and $E(v_1') = E(v_2')$. Using that fact and plugging in $E(v_3) = 0$ into the three equations above gives us:

$$E(v_1) = 1 + \frac{1}{3}E(v_1) + \frac{1}{3}E(C)$$

$$E(C) = 1 + \frac{2}{3}E(v_1')$$

$$E(v_1') = 1 + \frac{1}{2}E(v_1).$$

The above is a system of three equations with three unknowns. Solving the system gives us that $E(v_1) = 14/5 = 2.8$.

13. You are given a 5×5 chessboard along with a 1 (1×1) tile and 8 "L" tiles (each L tile taking 3 squares). How many squares on the chessboard can you place the 1×1 tile and still completely tile the remaining chess board.

    (a) 1

    (b) 5

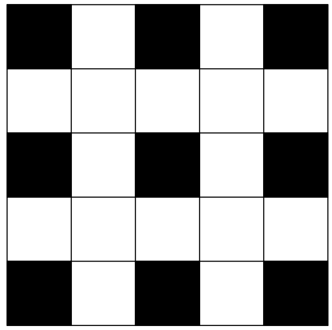
    (c) 9

    (d) 13

    (e) 25

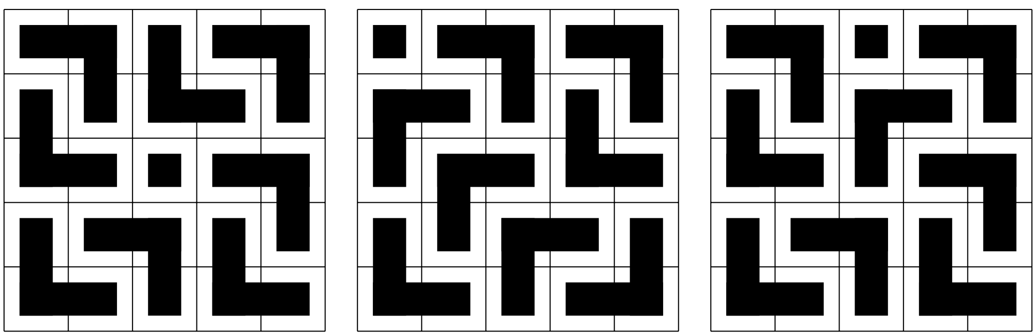
Solution: C: 9.

Notice that each of the 8 "L" tiles can only cover at most one of the 9 black squares pictured below



So the 1×1 tile must go in one of these positions. This gives an upper bound of 9.

To see that each of the 9 squares are viable, we construct three tilings which represent solutions to all 9 possibilities up to symmetry.



14. You are given a rod of unit length. Make two uniform independent perpendicular cuts on the rod to obtain three segments. What is the probability that the three segments form a triangle.

    (a) $\frac{1}{4}$
    (b) $\frac{1}{3}$
    (c) $\frac{2}{5}$
    (d) $\frac{1}{2}$
    (e) 1

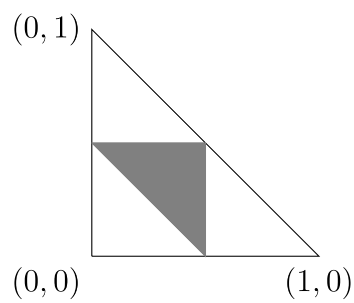
Solution: A: 1/4

Let a, b, c, be the lengths of each segment. Then (a, b) is uniformly distributed over the triangle $a \geq 0, b \geq 0, a + b \leq 1$ and $c = 1 - a - b$. For this to form a triangle, we need $a + b \geq c$, $a + c \geq b$ and $b + c \geq a$.

Using the identity $c = 1 - a - b$, this gives the inequalities

$$a + b \geq \frac{1}{2}$$
$$b \leq \frac{1}{2}$$
$$a \leq \frac{1}{2}$$

This region has 1/4-th of the total area. Also see the following picture.



15. Let n be the number of ways to fill a 10×20 grid with 0s and 1s such that every column and every row each have an odd number of ones. Find $\log_2(n)$.

    (a) 77
    (b) 91
    (c) 100

(d) 143

(e) 171

Solution: E: 171 Every such grid can be obtained uniquely by filling up the first 9×19 squares arbitrarily. So $n = 2^{9\cdot 19}$ and $\log_2(n) = 9 \cdot 19 = 171$.

To see this, observe that we can simply add a 0 or one to the bottom or to the right of the 9×19 grid to ensure that the corresponding column or row sums to 1 mod 2. We will check the bottom-most row and the right-most row last.

Explicitly, if the numbers we put in are $a_{i,j}$ then

$$a_{10,j} = 1 + \sum_{i=1}^{9} a_{i,j},$$
$$\text{and } a_{i,20} = 1 + \sum_{j=1}^{19} a_{i,j}$$

where everything is mod 2.

In order for this to work, we simply need to check that

$$\sum_{j=1}^{19} a_{10,j} = \sum_{i=1}^{9} a_{i,20}$$

so that the same correction bit goes into the bottom right-most corner. This is clear.

16. The Thue-Morse sequence is the following infinite sequence of binary digits:

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	⋯
0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1	0	0	⋯

It is formed by starting with 0, then duplicating the current sequence but flipping the bits in the second copy (0 to 1 and 1 to 0). So $0, 01, 0110, 01101001$, and so on. Find the 6 binary digits of this sequences beginning at 2019.

(a) 100110

(b) 100101

(c) 011010

(d) 001011

(e) 010011

Solution: E: 010011 Let t_n be the n-th term of the sequence. Then we have $t_0 = 0$ and given the finite string $t_0, \cdots, t_{2^n-1}$ of length 2^n, we get the next terms in the sequence as

$$t_{2^n+0} = 1 - t_0, \cdots, t_{2^{n+1}-1} = 1 - t_{2^n-1}.$$

That is, $t_{2^n+m} = 1 - t_m$ for all m between 0 and 2^n.

If we think about writing k in binary, then we see that we flip between 0 and 1 for each 1 that appears in the base-2 representation of k. That is, t_k is 0 if k has an even number of '1's in binary and 1 if k has an odd number of '1's.

Since $2019_{10} = 11111100011_2$. The 6 terms of this sequence beginning at 2019 are 010011 (the same sequence as begins at $3_{10} = 11_2$).

17. Let $\theta_1, \theta_2, \theta_3$ be real numbers satisfying $\cos 2\theta_1 = \cos 2\theta_2 = \cos 2\theta_3 = -\frac{7}{25}$. Find the largest possible value of $\cos(\theta_1 + \theta_2 + \theta_2)$.

   (a) $\frac{47}{64}$
   (b) $\frac{109}{126}$
   (c) $\frac{57}{64}$
   (d) $\frac{117}{125}$
   (e) 1

Solution: D: $\frac{117}{125}$

First, the cosine double angle formula $\cos 2x = 2\cos^2 x - 1$ gives us the equivalent conditions $\cos\theta_i = \pm\sqrt{\frac{\cos 2x+1}{2}} = \pm\frac{3}{5}$ for each $i = 1, 2, 3$. Denote $\theta = \arccos(\frac{3}{5}) \in (0, \frac{\pi}{2})$. We have $\cos\theta_i = \frac{3}{5}$ if and only if $\theta_i \equiv \theta$ or $-\theta \bmod 2\pi$. Also, we have $\cos\theta_i = -\frac{3}{5}$ if and only if $\theta_i = \pi - \theta$ or $\pi + \theta \bmod 2\pi$. In summary, the given condition is equivalent to $\theta_i = \theta, -\theta, \pi - \theta,$ or $\pi + \theta \bmod 2\pi$ for each i. Or in other words, $\theta_i = \theta$ or $-\theta \bmod \pi$ for each i. Therefore, their sum is $\theta_1 + \theta_2 + \theta_3 = -3\theta, -\theta, \theta,$ or $3\theta \bmod \pi$. Using the fact that $\cos x = \pm\cos y$ if $x \equiv y \bmod \pi$ (which in turn follows from the identity $\cos(x + \pi) = -\cos x$), we find that $\cos(\theta_1 + \theta_2 + \theta_3)$ is equal to $\pm\cos(-3\theta)$, $\pm(-\theta)$, $\pm\cos\theta$, or $\pm\cos 3\theta$. Since cos is an even function, we can say more simply that $\cos(\theta_1 + \theta_2 + \theta_3)$ is equal to $\pm\cos 3\theta$ or $\pm\cos\theta$.

Now we actually compute these values. We already know from the definition of θ that $\cos\theta = \frac{3}{5}$. The cosine triple angle formula then yields $\cos 3\theta = 4\cos^3\theta - 3\cos\theta = -\frac{117}{125}$. Thus $\cos(\theta_1 + \theta_2 + \theta_3)$ is equal to $\pm\frac{3}{5}$ or $\pm\frac{117}{125}$, so it is at most $\frac{117}{125}$. Moreover, observe that this is value is achieved when, for example, $\theta_1 = \theta_2 = \theta$ and $\theta_3 = \pi + \theta$, since in this case $\cos(\theta_1 + \theta_2 + \theta_3) = \cos(\pi + 3\theta) = -\cos 3\theta = \frac{117}{125}$. We conclude that the answer is $\frac{117}{125}$.

18. Let $ABCD$ be a convex quadrilateral. If $AB = 5$, $CD = 7$, $\angle B = 45°$, $\angle C = 60°$, and $\angle D = 90°$, compute AD.

   (a) $5\sqrt{5} - 3\sqrt{2}$
   (b) $7\sqrt{3} - 5\sqrt{2}$
   (c) $5\sqrt{5} + 3\sqrt{2}$
   (d) $7\sqrt{3} + 5\sqrt{2}$
   (e) 9

Solution: B: $7\sqrt{3} - 5\sqrt{2}$

Place the quadrilateral in the first quadrant of the Cartesian plane, with $B = (0, 0)$ and $C = (c, 0)$, say. By dropping perpendiculars from A and D to the x-axis BC, we see that $A = (5\cos B, 5\sin B) = (\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2})$ and $D = (c - 7\cos C, \sin C) = (c - \frac{7}{2}, \frac{7\sqrt{3}}{2})$. Next, let E be the foot of the perpendicular from A to the vertical line through D. Since the y-coordinate of D is higher than that of A, and $\angle A = 360^\circ - \angle B - \angle C - \angle D = 360^\circ - 45^\circ - 60^\circ - 90^\circ = 165^\circ$, we have $\angle DAE = \angle A - \angle BAE = 165^\circ - (45^\circ + 90^\circ) = 30^\circ$. Also, $DE = \frac{7\sqrt{3}}{2} - \frac{5\sqrt{2}}{2}$, so from right triangle ADE we get $AD = \frac{DE}{\sin \angle DAE} = \frac{\frac{7\sqrt{3}}{2} - \frac{5\sqrt{2}}{2}}{\sin 30^\circ} = 7\sqrt{3} - 5\sqrt{2}$.

19. Calculate the (shaded) area of the Hawaiian earring space, which is depicted in Figure 3. This space is built by taking a circle of radius 1, removing a circle of radius $\frac{1}{2}$, adding a circle of radius $\frac{1}{3}$, removing a circle of radius $\frac{1}{4}$, and so on.

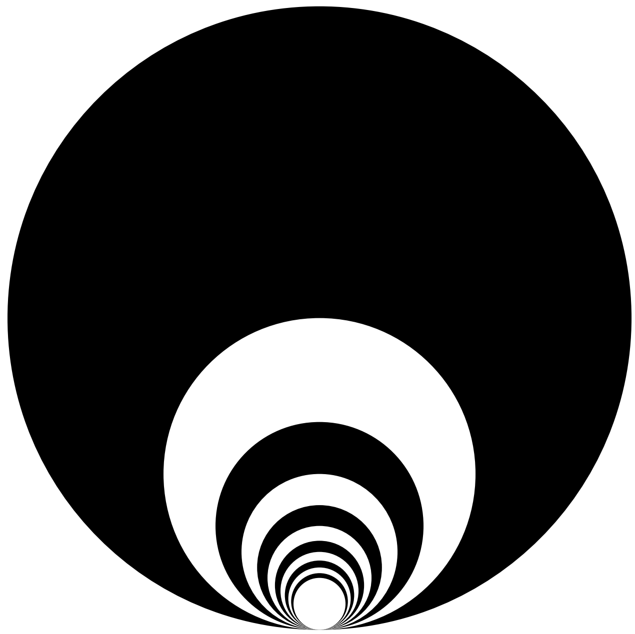


Figure 3: Hawaiian earring space

(a) $\frac{3\pi}{4}$

(b) $\frac{4\pi}{5}$

(c) $\frac{\pi^3}{12}$

(d) $\frac{4\pi^2}{15}$

(e) $\frac{\pi^3}{15}$

Solution: C: $\dfrac{\pi^3}{12}$.

The area is given by

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^{n+1}\pi\left(\frac{1}{n}\right)^2 &= \pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\\ &= \pi\cdot\eta(2),\end{aligned}$$

where $\eta(s)$ is the Dirichlet eta function. Note that

$$\begin{aligned}\zeta(s)-\eta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s}-\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^s}\\ &= \sum_{n=1}^{\infty}\frac{2}{(2n)^s}\\ &= 2^{1-s}\sum_{n=1}^{\infty}\frac{1}{n^s}\\ &= 2^{1-s}\zeta(s).\end{aligned}$$

Thus $\eta(s)=(1-2^{1-s})\zeta(s)$. Since

$$\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},$$

it follows that $\eta(2)=(1-2^{-1})\frac{\pi^2}{6}=\frac{\pi^2}{12}$. As a result, the area is $\frac{\pi^3}{12}$.

20. Let ABC be an acute triangle with $AB=3$ and $AC=4$. Denote the circumcenter of ABC by O, and let line AO intersect BC at D. Given that $BD/DC=3$, find the **square** of the circumradius of ABC.

    (a) $\frac{247}{60}$

    (b) $\frac{62}{15}$

    (c) $\frac{251}{60}$

    (d) $\frac{64}{15}$

    (e) $\frac{257}{60}$

Solution: A: $\frac{247}{60}$

It will be convenient to first establish a few simple lemmas.

Lemma 1. We have $\angle DAB = 90^\circ - C$ and $\angle DAC = 90^\circ - B$.

Proof. Denote $x=\angle OAB=\angle OBA$, $y=\angle OBC=\angle OCB$, and $z=\angle OCA=\angle OAC$. Then by looking at each vertex of triangle ABC we obtain $x+z=A$, $x+y=B$, and $y+z=C$. Adding gives $2x+2y+2z=A+B+C=180^\circ$, so $x+y+z=90^\circ$ and thus $\angle DAB = x = 90^\circ$ and $\angle DAC = z = 90^\circ - B$, as desired. □

Lemma 2. ("Ratio Lemma") We have $\frac{BD}{DC} = \frac{AB}{AC}\frac{\sin\angle DAB}{\sin\angle DAC}$.

Proof. Applying the Law of Sines to triangles DAB and DAC yields $\frac{BD}{\sin\angle DAB} = \frac{AB}{\sin\angle ADB}$ and $\frac{DC}{\sin\angle DAC} = \frac{AC}{\sin\angle ADC}$, respectively. But $\angle ADB + \angle ADC = 180°$, so $\sin\angle ADB = \sin\angle ADC$. Rearranging and and then dividing the two equations now gives the desired result. □

Using the given information and Lemma 1 and Lemma 2, we can write

$$3 = \frac{BD}{DC} = \frac{AB}{AC}\frac{\sin\angle DAB}{\sin\angle DAC} = \frac{3}{4}\frac{\sin(90° - C)}{\sin(90° - B} = \frac{3}{4}\frac{\cos C}{\cos B},$$

so $\frac{\cos C}{\cos B} = 4$. In addition, the Law of Sines on triangle ABC gives $\frac{\sin C}{\sin B} = \frac{AB}{AC} = \frac{3}{4}$. It remains only to solve this system of two equations for B and C.

Squaring both equations gives $\frac{1-\sin^2 C}{1-\sin^2 B} = 16$ and $\frac{\sin^2 C}{\sin^2 B} = \frac{9}{16}$, respectively. Substituting $\sin^2 C = \frac{9}{16}\sin^2 B$ from the second equation into the first and solving for $\sin^2 B$ yields $\sin^2 B = \frac{240}{247}$. Finally, the Extended Law of Sines gives the square of the circumradius of ABC to be

$$R^2 = \frac{AC^2}{4\sin^2 B} = \frac{4^2}{4 \cdot \frac{240}{247}} = \frac{247}{60}.$$

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