Georgia Tech High School Mathematics Competition
Group Test
April 13, 2019

Instructions: This exam consists of eight questions. You will have 90 minutes to complete as much of the exam as possible. You may collaborate only within your group on this test. Do not use outside notes or electronic devices. Write your names and answers on the provided answer sheet.

1. Suppose that a bag has 7 blue balls, 8 red balls, and 9 yellow balls. If three balls are drawn from the bag, what is the probability that they are all different colors?

Solution: If we number the balls 1 through 24, then there are (24)(23)(22) ways to draw the ball from the bag if order matters. Then, if all 3 are different colors, we have 7 choices for the blue ball, 8 choices for the red ball, 9 choices for the yellow ball, and 6 ways to order them. So, the probability is

$$\frac{7 * 8 * 9 * 6}{24 * 23 * 22} = \frac{63}{253}$$

2. On a table, you have 11 stones of identical weight, and 1 stone that is either lighter or heavier than each of the other 11. Using a scale at most 3 times, determine which stone is different. Can you tell whether it is heavier or lighter than the other stones?

Solution: Label the stones A,B,C,D,E,F,G,H,I,J,K,L. Using the scale, weigh stones A,B,C,D against stones E,F,G,H. **(First use).**

Case 1: If the scale is level, then the odd stone is one of I,J,K,L. Weigh I,J against A,B. **(Second use.)**

Case 1.1: If the scale is level, then the odd stone is one of K,L. Weigh K against A. **(Third use.)** If the scale tips, then K is the odd stone, and K is heavy if it tips down and light if it tips up. Otherwise, L is the odd stone, and we do not know if it is heavy or light.

Case 1.2: If the scale tips, then the odd stone is one of I,J. If the scale tips down, the odd stone is heavy, and if it tips up, the odd stone is light. Weigh I against A. **(Third use.)** If the scale tips, then I is the odd stone. Otherwise, J is the odd stone.

Case 2: If A,B,C,D are heavier than E,F,G,H, then the odd stone is one of these 8. Weigh I,J,K,H against E,F,G,D. **(Second use.)**

Case 2.1: If the scale is now level, then the odd stone was one of A,B,C, and it is heavier than a normal stone. Weigh A against B. **(Third use.)** If the scale tips, the heavier stone is the odd stone. Otherwise, C is the odd stone.

Case 2.2: If the scale shows that I,J,K,H are heavier than E,F,G,D, then the odd stone was one of E,F,G, and the odd stone is light. Weigh E against F. (**(Third use.)** If the scale tips, then the lighter stone is the odd one. Otherwise, G is the odd stone.

Case 2.3: If the scale shows that I,J,K,H are lighter than E,F,G,D, then the odd stone is either D or H. Weigh A against D. **(Third use.)** If the scale tips, then D is the odd stone, and we know if it is light or heavy. Otherwise, H is the odd stone, and it is light.

Case 3: If A,B,C,D are lighter than E,F,G,H, then proceed as in cases 2.1, 2.2, and 2.3, this time simply swapping the names A with E, B with F, C with G, and D with H.

3. Alice and Bob play a game where they first collectively agree on a three letter word. Then, Alice writes on individual pieces of paper every single possible way to rearrange the letters in that word (excluding the original arrangement of letters). All of these pieces of paper are put into a hat and then one of these pieces are drawn uniformly at random. Bob then takes what is written on the chosen piece of paper and transcribes it onto a whiteboard. Both players then take turns picking two letters on the whiteboard and switching their positions. The first player to switch the letters in a way that forms the original 3-letter word wins. There is an additional rule that the next player cannot "undo" the turn of the previous player (e.g. if Alice switches the letters in positions 1 and 2, then in Bob's next turn, he may not switch the letters in positions 1 and 2). Alice always goes first. Assuming both players play perfectly, what is the probability that Alice will win or draw? (A draw occurs if both players are able to play the game indefinitely.)

Solution: First, we consider all the possible ways the letters can be arranged. There are $6! - 1 = 5$ ways to arrange the letters excluding the original word. We can split the arrangements based on the form of their disjoint cycle decompositions. There are 3 arrangements that are a product of a 2-cycle and 1-cycle and there are two arrangements of the form of a 3-cycle. We consider each case separately. Without loss of generality, suppose the original word is "abc".

We consider the first case where the rearrangement is just the result of a transposition of two letters of the original word such as "bac". In this case, Alice can use her first turn to apply the same transposition (swapping 'a' and 'b') to get back the original turn. So Alice wins in this case assuming she plays perfectly.

Next, we consider the second case where the rearrangement has the form of a 3-cycle such as "bca". Alice's only options are to transform "bca" into either "bac", "cba", or "acb". In any of those three cases, Bob has the ability to swap two letters in a way to get back to the original word. Thus, Bob will win if the rearrangement of the letters has the form of a 3-cycle.

Thus, the probability Alice will win is the probability that the piece of paper pulled out of the hat is the result of a transposition of two letters of the original word which we know happens 3/5 times. There is no way for the game to end in a draw if both players play optimally so the probability of Alice either winning or drawing is $3/5 + 0 = 3/5$.

4. Suppose a clock has that at any given time its second hand is 5 seconds fast and its minute hand is 15 minutes fast. Find a value c such that if the hour hand is always c hours fast, then at some time t the second, minute, and hour hands are all aligned.

Solution: Let x_0 be the position of the second hand, x_1 be the position of the minute hand, and x_2 the position of the hour hand in terms of seconds/minutes around the clock. Let t be in minutes and suppose that the hour hand is c hours fast. Then,

$$\begin{cases} x_0(t) = 60t + 5 \mod 60 \\ x_1(t) = t + 15 \mod 60 \\ x_2(t) = \frac{t}{12} + 5c \mod 60 \end{cases}$$

The times t such that $x_0(t) = x_1(t)$ will have that $60t + 5 = t + 15 \mod 60$ which implies that $59t = 10 \mod 60$. This will hold if and only if $t = \frac{10+60k}{59}$ for some integer k. Then, setting $x_1 = x_2$ we get that $5c = \frac{11}{12}t + 15 = \frac{11}{12}(\frac{10+60k}{59}) + 15 \mod 60$. Converting this from minutes around the clock to hours, we get that $c = \frac{11}{60}(\frac{10+60k}{59}) + 3 = \frac{11+66k}{(6)(59)} + 3 \mod 12$. The choices of k give all possible solutions to the problem.

5. Suppose that a graph G is planar (can be drawn on a plane with no crossing edges) is bipartite (the vertices can be divided into two sets, V_1 and V_2 such that every edge in G is between a vertex in V_1 and a vertex in V_2) and has at least 8 faces (i.e, it divides the plane into at least 8 regions). What is the smallest number of edges that G could have?

Solution: 16. Note that the stereographic projection of the cube gives a bipartite planar graph with 16 edges and 8 faces, so all that remains is to show that this example is minimal. Note that since G is bipartite, it must be triangle-free, so every face has at least 4 vertices. Then the number of edges is at least

$$\frac{\text{(number of faces)(edges per face)}}{\text{(number of times each edge is counted)}} \geq \frac{(8)(4)}{2} = 16.$$

This shows no such graph could have fewer edges.

6. Is it possible to cut up a square into triangles such that each triangle has only acute angles? If so, what is the minimum required number of triangles to do so?

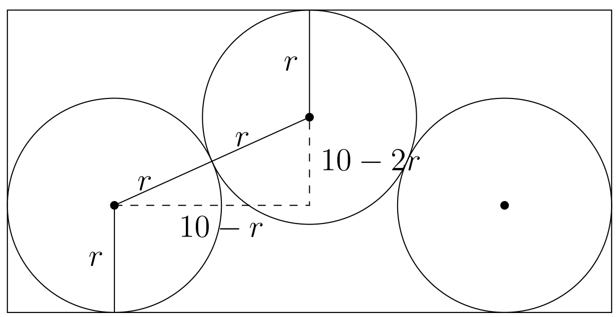
Solution: It can, and requires 8 triangles. This can be found by observing that every internal vertex must be adjacent to at least 5 triangles, every vertex on the edge to at least 3, and every corner to at least 2. Then, the number of triangles is

$$\frac{2(\#\text{corner vertices}) + 3(\#\text{edge vertices}) + 5(\#\text{internal vertices})}{3} = \frac{8 + 3k + 5j}{3}$$

There must be at least 2 internal vertices for the above to be an integer. When trying to draw it, you find the need for at least 2 edge vertices. Then, there are $\frac{8+6+10}{3} = 8$ triangles. With a little more time I can probably make this more precise.

7. You are given a 10×20 sheet of material from which you need to cut out three circles of equal radius. Find with proof, the largest possible radius.

Solution: The optimal arrangement is as follows.



By the Pythagorean Theorem, we have $(10-r)^2 + (10-2r)^2 = (2r)^2$. Solving gives

$$\begin{aligned} (10-r)^2 + (10-2r)^2 &= (2r)^2 \\ 100 - 20r + r^2 + 100 - 40r + 4r^2 &= 4r^2 \\ r^2 - 60r + 200 &= 0. \end{aligned}$$

Since we need $r < 10$, we take the root of this with the minus sign in front of the square root:

$$\begin{aligned} r &= \frac{60 - \sqrt{3600 - 800}}{2} \\ &= 10(3 - \sqrt{7}) \end{aligned}$$

To see that this is optimal, we show that if $R > r$ is any larger radius, then we cannot place three circles of radius R into the rectangle.

To see this, let the triangle be the set $[0, 20] \times [0, 10]$ and note that the centers of the circles must lie inside the set $[R, 20 - R] \times [R, 10 - R]$.

The left half of this set is $[R, 10] \times [R, 10 - R]$ and this has a diagonal length of

$$\sqrt{(10 - R)^2 - (10 - 2R)^2} < 2r < 2R.$$

In particular, this half of the set can contain at most one center of a circle without overlapping the circles. Therefore we can place at most two circles (one centered in each half).

8. 1. Investigate all the possible dollar values that can be obtained using a \$3 coin and a \$5 coin. Possible hint: $2 \cdot 3 - 1 \cdot 5 = 1$ and $2 \cdot 5 - 3 \cdot 3 = 1$.
   2. Your local cookie store sells cookies in packs of 6, 10 and 15. Which numbers of cookies is it possible to order?
   3. Develop a general strategy for answering such problems.
      (a) When is every number a possible combination?
      (b) When are there finitely many impossible combinations?
      (c) If there are only finitely many impossible combinations, can you find a bound N (depending on the input) such that every number $n \geq N$ is a possible combination?
      (d) Can you find an exact bound or only an approximate bound?
      (e) If you can't find an exact bound in the most general case, can you find one if there are only 2 input values?

Solution:

1. $1, 2, 4, 7$ are impossible. All other values are possible.
2. $1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23, 29$ are impossible. All other values are possible.
3. If the inputs are $a_1, \ldots, a_k$ and $\gcd\{a_1, \ldots, a_k\} = 1$ then there will only be finitely many non-solutions. If the gcd is not 1 then any number which is not a multiple of the gcd is impossible.

   The largest number such that $a_1x_1 + \cdots + a_kx_k = n$ has no solution is called the Frobenius number. Several bounds are known. See e.g.

   - *The Diophantine Frobenius Problem* by J. L. Ramírez Alfonsín (Oxford University Press, 2005)
   - or Dixmier, J., *Proof of a conjecture by Erdős and Graham concerning the problem of Frobenius*, Journal of Number Theory, 34 (1990), 198–209

- or https://cs.uwaterloo.ca/˜shallit/Talks/frob6.pdf
- or https://www.emis.de/journals/PM/55f4/pm55f406.pdf

One crude bound is $\max\{a_1, \ldots, a_k\}^2$. A more refined estimate conjectured by Erdős and Graham and proved by Dixmier is $\max\{a_1, \ldots, a_k\}^2/(k-1)$.

For just two inputs a, b with $\gcd(a, b) = 1$, we have the bound $ab - a - b$ due to Sylvester.

Proof. First, we show that $ab - a - b$ is impossible. To that end, suppose $ab - a - b = ax + by$ for some non-negative integers x and y. Then looking modulo a and b,

$$\begin{aligned} -b &\equiv by \pmod{a} \\ -a &\equiv ax \pmod{b} \end{aligned}$$

So $x \equiv -1 \pmod{b}$ and $y \equiv -1 \pmod{a}$. Since x and y are non-negative, this means $x \geq (b-1)$ and $y \geq (a-1)$. But now

$$ax + by \geq a(b-1) + b(a-1) = 2ab - a - b > ab - a - b,$$

which is impossible.

Now let $n \geq ab - a - b + 1 = (a-1)(b-1)$. By the extended gcd algorithm, there exist integers u, v such that $au + bv = 1$ and hence $aun + bvn = n$. Moreover, every solution to $ax + by = n$ is given by

$$a(x + kb) + b(y - ka) = n$$

for some integer k.

Among all these solutions, (x, y), pick the one where x is non-negative and as small as possible. Then $x \leq b - 1$ since otherwise $(x - b, y + b)$ has $x - b$ smaller and non-negative.

Now

$$\begin{aligned} by &= n - ax \\ &\geq (a-1)(b-1) - ax \\ &\geq (a-1)(b-1) - a(b-1) \\ &\geq (a-1-a)(b-1) \geq 1 - b. \end{aligned}$$

So by is a multiple of b which is at least $-b + 1$ and therefore $by \geq 0$. I.e. $y \geq 0$.

This shows that $ax + by = n$ has a solution where x, y are non-negative. $\square$

End of exam.