High School Math Competition – 2021
Proof Exam Solutions

Georgia Tech School of Math

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1. An ant is moving around a rubber band at a constant speed of 1 cm/s. The rubber band starts at 1 m in circumference and every second, the rubber band instantaneously grows in circumference by 1 m. Does the ant ever make a full lap? Prove or disprove.

Solution: Yes. In the first second, the ant travels 1% around the rubber band. Then $\frac{1}{2}$%, then $\frac{1}{3}$% and so on. So the question is: is there a number n such that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots + \frac{1}{n} \geq 100?$$

By a crude estimation, we can see that

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots \geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\cdots = 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots$$

and this will be ≥ 100 once we take enough terms.

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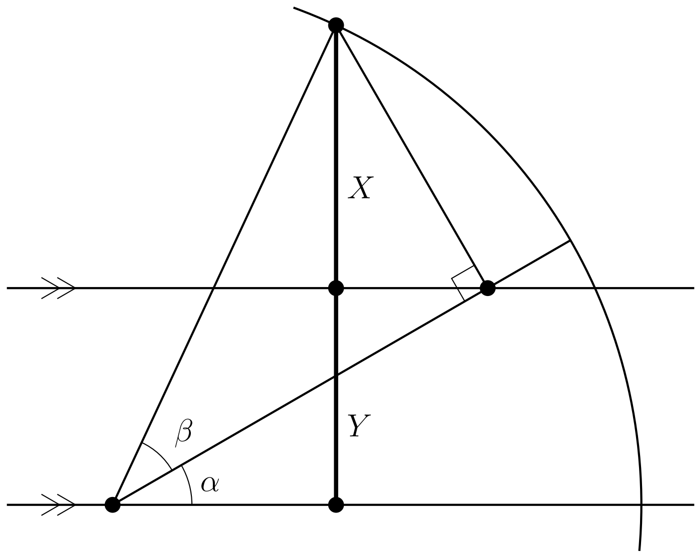
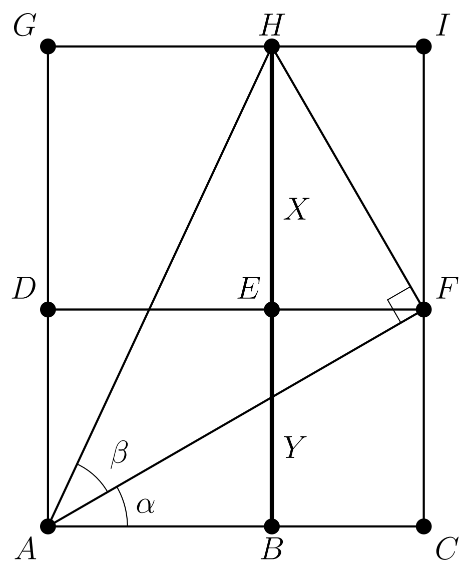
2. In the following diagram, angles α and β are both no more than 45°, the two horizontal lines are parallel and the triangle is right. Prove that if $\alpha = \beta$ then the lengths of the vertical segments, X and Y, are equal. Conversely, show that if $X = Y$ then $\alpha = \beta$.



Solution: First, we show that $X = \cos\alpha \sin\beta$ and $Y = \sin\alpha \cos\beta$. For this it helps to redraw the diagram as follows:



Let us take the radius of our circle to be $AH = 1$. Then $AF = \cos\beta$ and hence $Y = CF = \sin\alpha \cos\beta$.

Likewise, we notice that $HF = \sin\beta$ and $\angle HFI = \alpha$. Thus $X = FI = \sin\beta \cos\alpha$.

Clearly, if $\alpha = \beta$ then $X = \sin\beta\cos\alpha = \sin\alpha\cos\beta = Y$.

Conversely, if $X = Y$ then

$$1 = \frac{X}{Y} = \frac{\sin\beta\cos\alpha}{\sin\alpha\cos\beta} = \frac{\tan\beta}{\tan\alpha}.$$

Thus $\tan\alpha = \tan\beta$, and because $0 \le \alpha, \beta \le 45°$, this implies that $\alpha = \beta$.

3. For every group of 10 people, every pair of people either know each other or don't. Prove that no matter what collection of pairs know each other, there will always be a set of 4 people in which all know each other or a set of three people in which nobody knows anybody else.

   You may use the fact that for any 6 people, there will always be a group of 3 which all know each other or nobody knows anybody.

Solution: Fix a person p. Split the remaining people into two sets:

$$K = \{\text{people who know } p\} \text{ and } N = \{\text{people who do not know } p\}.$$

Notice that either $|K| \geq 6$ or $|N| \geq 4$.

In the first case, if $|K| \geq 6$ then there are three people in K which all know each other or all don't know each other. In both cases we are done because either we have 3 people which do not know each other or 3 people which all know each other and also know p.

If $|N| \geq 4$ then either all of them know each other, in which case we are done, or there is a pair $\{x, y\}$ which do not know each other. In this last case $\{x, y, p\}$ all do not know each other.

So in all cases we have found a set of 4 people which all know each other or 3 people which all do not.

4. Derive a formula for the number of ways to color nodes labeled $\{1, \ldots, n\}$ using t colors such that for every i, nodes i and $i+1$ have different colors and also 1 and n have different colors.

Solution: Let $f_n(t)$ denote this number.

First solution (Deletion/Contraction): If we apply deletion-contraction. There are $t(t-1)^{n-1}$ ways to color $\{1, \ldots, n\}$ such that i and $i+1$ have different colors (considered $mod n$). Of these, $f_{n-1}(t)$ will color 1 and n the same. Therefore, we have the recurrence

$$f_n(t) = t(t-1)^{n-1} - f_{n-1}(t).$$

Writing this out, we obtain

$$\begin{aligned} f_n(t) &= t(t-1)^{n-1} - t(t-1)^{n-2} + t(t-1)^{n-3} + \cdots + (-1)^{n-2} t(t-1) \\ &= t(t-1) \sum_{k=0}^{n-2} (t-1)^k (-1)^{n-2-k} \\ &= (-1)^{n-2} t(t-1) \frac{1-(1-t)^{n-1}}{1-(1-t)} \\ &= (t-1)^n + (-1)^n (t-1). \end{aligned}$$

Second solution (Inclusion/Exclusion): For each subset $S \subseteq \{1, \ldots, n\}$, let A_S be the set of valid colorings such that j is colored the same as $j-1 \bmod n$ for all $j \in S$. By the Principle of Inclusion and Exclusion,

$$f_n(t) = |A_\varnothing| - \sum_i |A_{\{i\}}| + \sum_{i<j} |A_{\{i,j\}}| - \sum_{i<j<k} |A_{\{i,j,k\}}| + \cdots .$$

Of course,

$$|A_S| = \begin{cases} t^{n-|S|} & \text{if } S \neq \{1, \ldots, n\}, \\ t & \text{if } S = \{1, \ldots, n\}. \end{cases}$$

Therefore,

$$\begin{aligned} f_n(t) &= \sum_{s=0}^{n} \sum_{|S|=s} (-1)^s t^{n-s} + \underbrace{(-1)^n (t-1)}_{\text{Correction term for s = n}} \\ &= \sum_{s=0}^{n} \binom{n}{s} (-1)^s t^{n-s} + (-1)^n (t-1) \\ &= (t-1)^n + (-1)^n (t-1). \end{aligned}$$

5. Let $f(x) = \sum_{i=0}^{m} c_i x^i$ be a polynomial with integer coefficients. Let $f'(x) = \sum_{i=0}^{m} i c_i x^{i-1}$. Suppose that f and f' do not have a common zero mod p. Show that if $f(x) \equiv 0 \pmod{p}$ has exactly k solutions, then $f(x) \equiv 0 \pmod{p^n}$ has at most k solutions for every n.

   Note: solutions are considered to be "the same" if they are congruent mod p^n. This means that if $a \equiv a' \pmod{p^n}$ then $f(a) \equiv f(a') \equiv 0$ counts as only one solution.

Solution: We proceed by induction with the base case ($n = 1$) being the hypothesis. Suppose $a_1, \dots, a_{k'}$ are the $k' \le k$ solutions mod p^n. If a is a solution mod p^{n+1} then a is a solution mod p^n hence $a \equiv a_j \pmod{p^n}$ for some j. We show that for each $1 \le j \le k'$ there is at most one solution a mod p^{n+1} such that $a \equiv a_j \pmod{p}$.

Suppose there is a pair of distinct solutions a, b such that $a \equiv b \equiv a_j \pmod{p^n}$. Then we have

$$\begin{aligned} 0 &\overset{p^{n+1}}{\equiv} \frac{f(b) - f(a)}{b - a} \\ &= \sum_{i=0}^{m} c_i \frac{b^i - a^i}{b - a} \\ &= \sum_{i=0}^{m} c_i \sum_{t=0}^{i-1} b^t a^{i-1-t} \\ &\overset{p^n}{\equiv} \sum_{i=0}^{m} c_i \sum_{t=0}^{i-1} a_j^{i-1} \\ &= \sum_{i=0}^{m} i c_i a_j^{i-1} \\ &= f'(a_j). \end{aligned}$$

But this is impossible since now $f(a_j) \equiv f'(a_j) \equiv 0 \pmod{p}$ which contradicts the hypothesis.

Thus for every a_j there is at most one a and hence there are at most $k' \le k$ solutions in total.

If one has seen Taylor's Theorem before, one can give an alternative proof using that. We apply Taylor's Theorem, centered at some root a mod p^n to get

$$f(x) = f(a) + f'(a)(x - a) + g(x)(x - a)^2$$

for some polynomial $g(x)$. And, we want to solve the system of equations

$$\begin{aligned} x &\equiv a \pmod{p^n}, \\ f(x) &\equiv 0 \pmod{p^{n+1}}. \end{aligned}$$

If $x \equiv a \pmod{p^n}$ then $x - a = tp^n$ for some integer t. And substituting this into what Taylor's Theorem told us, we have

$$f(x) = f(a) + f'(a)tp^n + g(x)t^2p^{2n}.$$

Because of that p^{2n}, we can therefore conclude that

$$f(x) \equiv f(a) + f'(a)tp^n \pmod{p^{n+1}}.$$

We can begin to solve for t:

$$f(x) \equiv 0 \pmod{p^{n+1}} \iff f(a) + f'(a)tp^n \equiv 0 \pmod{p^{n+1}}.$$

Also, $f(a) \equiv 0 \pmod{p^n}$ so $f(a) = sp^n$ for some integer s. Thus we can write

$$sp^n + f'(a)tp^n \equiv 0 \pmod{p^{n+1}} \iff s + f'(a)t \equiv 0 \pmod{p}$$

and solving for t gives

$$t \equiv -s[f'(a)]^{-1} \pmod{p},$$

which is sensible because $f'(a) \not\equiv 0$ by hypothesis.

Therefore there is a unique solution to the system of equations, given by

$$x = a + tp^n, \text{ where } t \equiv -s[f'(a)]^{-1} \pmod{p}.$$

So we get a stronger result that there are exactly k solutions mod p^n for all n using the above to lift solutions from p^n to p^{n+1}.

In fact, $tp^n \equiv -sp^n[f'(a)]^{-1} \equiv -f(a)[f'(a)]^{-1} \pmod{p^{n+1}}$. Some more advanced students will recognize this as Newton's method:

$$x \equiv a - \frac{f(a)}{f'(a)} \pmod{p^{n+1}}.$$

6. A k-term arithmetic progression or k-AP is a sequence $a, a+d, a+2d, \ldots, a+(k-1)d$ of length k.

Fix a k-AP in $\{1, \ldots, n\}$ and find an upper bound on the number of other k-APs in $\{1, \ldots, n\}$ that share at least one term with the fixed k-AP. You may assume that k is even or k is odd if it helps simplify your calculations.

Solution: We can show that $k(n-1)$ is an upper bound when k is even.

This follows from the following claim: for a fixed $x \in \{1, \ldots, n\}$, there are at most $n-1$ arithmetic progressions that have x as a term.

To see this, observe that if x appears as the i-th term of the AP, then $x = a+(i-1)d$ and we must have $a = x-(i-1)d \geq 1$ and $a+(k-1)d = x+(k-i)d \leq n$. Thus

$$d \leq \min\left\{\frac{x-1}{i-1}, \frac{n-x}{k-i}\right\}.$$

The first bound is better for $i > k/2$ and the second is better for $i \leq k/2$. Thus we have

$$\begin{aligned}
\#k\text{-APs that have } x \text{ as a term} &\leq \sum_{i=1}^{k/2} \frac{n-x}{k-i} + \sum_{i=k/2+1}^{k} \frac{x-1}{i-1} \\
&= ((n-x)+(x-1)) \sum_{i=k/2+1}^{k} \frac{1}{i-1} \\
&\leq (n-1) \sum_{i=k/2+1}^{k} \frac{1}{k/2} \\
&= (n-1),
\end{aligned}$$

which proves the claim.