High School Math Competition – 2021
Multiple Choice Solutions

Georgia Tech School of Math

April 24, 2021

1. Write $157 = a^2 + b^2$ for some positive integers a, b. What is $a + b$?

> **Solution: 17**.
>
> $157 = 6^2 + 11^2$ so $a + b = 17$.

2. Find the remainder when 2^{75} is divided by 15.

> **Solution: 8**.
>
> Since $2^4 = 16 \equiv 1 \pmod{15}$, we have that $2^{75} \equiv 2^{4\cdot 18+3} \equiv 1^{18} \cdot 2^3 \pmod{15}$. So, $2^{75} \equiv 8 \pmod{15}$.

3. Let

$$a_1 = 1, a_2 = 1 + \frac{1}{1}, a_3 = 1 + \frac{1}{1 + \frac{1}{1}}, a_4 = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}$$

and continued in the same manner. Find a_{10}.

Problems and solutions by:
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Solution: $\mathbf{\frac{89}{55}}$.
We note that $a_1 = \frac{1}{1}, a_2 = \frac{2}{1}, a_3 = \frac{3}{2}, a_4 = \frac{5}{3}$ and, in general, if $a_n = \frac{p}{q}$ then $a_{n+1} = \frac{p+q}{p}$. We can prove this via the identity $a_{n+1} = 1 + \frac{1}{a_n}$.
In any event, we observe that the numerator and denominator are Fibbonacci numbers and just carry out this pattern to a_{10}:

$$a_5 = \frac{8}{5}, a_6 = \frac{13}{8}, a_7 = \frac{21}{13}, a_8 = \frac{34}{21}, a_9 = \frac{55}{34}, a_{10} = \frac{89}{55}$$

4. What is the area of a rectangle whose width is 2 and which is inscribed in a circle of radius 5.

Solution: $\mathbf{8\sqrt{6}}$.
The diagonal of the rectangle is the diameter of the circle, which is 10. Therefore, by the Pythagorean Theorem, the height of the rectangle is $\sqrt{10^2 - 2^2} = \sqrt{96}$, and the area is

$$2\sqrt{96} = 8\sqrt{6}.$$

5. How many ways are there to put balls numbered $1, 2, 3, 4$ into bins numbered $1, 2, 3, 4$ such that each bin has exactly one ball and no ball is placed into the bin of the same number?

Solution: **9**.
We can list these explicitly:

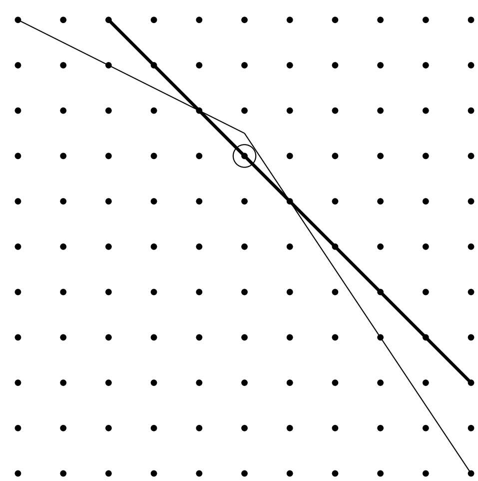
2143
2341
2413
3142
3412
3421
4123
4312
4321

Alternatively, such a permutation either arises by shifting the balls $1, 2$ or 3 places in a cycle, or is obtained by picking two pairs and swapping the positions of the balls in those pairs (e.g. swap 1 and 2 and swap 3 and 4). Thus there are $3 + \binom{4}{2} = 9$ such permutations.

6. What is the maximum value of $x + y$ where x and y are non-negative integers satisfying $x + 2y < 20$ and $3x + 2y < 30$.

Solution: 12.
First note that if we add the two constraints, we get $(x+2y)+(3x+2y) = 4(x+y) < 50$. Thus $x + y < 12.5$. Also if we solve $x + 2y = 20$ and $3x + 2y = 30$, we get $(x, y) = (5, 7.5)$ from which we might guess that $(5, 7)$ is the optimal *integer-valued* solution. And indeed, because $x+y$ can be no more than 12.5 and $5+7 = 12$, there can be no integer-valued solution with a larger value of $x + y$.

It is also possible to work this out graphically.



The value c of $x + y$ is maximized when the line $x + y = c$ is furthest to the right.

7. There is exactly one degree 4, monic polynomial p with integer coefficients such that $p(\sqrt{2} + \sqrt{5}) = 0$. What is the sum of the absolute values of the coefficients of p?

Note: "monic" means that the highest power of x has a coefficient of 1.

Solution: 24.
Let $x = \sqrt{2} + \sqrt{5}$. Then $x^2 = 7 + 2\sqrt{10}$ so $(x^2 - 7)^2 = 40$. If we expand this, we obtain

$$x^4 - 14x^2 + 9 = 0.$$

8. Evaluate $\sum_{k=0}^{15} \binom{15-k}{k}$.

Solution: 987.
Computing the sequence

$$a_n = \sum_{k=0}^{n} \binom{n-k}{k}$$

for small n is enough to guess that $a_n = F_{n+1}$ is the $(n+1)$-st Fibonacci number. We want $F_{16} = 987$.

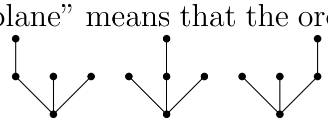
To explain why these are Fibonacci numbers, note that a_n counts the number of ways to write n as a sum of 1's and 2's where order matters. Here k counts the number of 2's.

E.g. for $n = 5$ we are looking at all permutations of $(1, 1, 1, 1, 1)$ (5 choose 0) plus all permutations of $(1, 1, 1, 2)$ (4 choose 1) plus all permutations of $(1, 2, 2)$ (3 choose 2).

On the other hand, we can decompose such a sum into a head and a tail: $(1 + 2 + 1 + 1 + 2) \leftrightarrow (1, (2 + 1 + 1 + 2))$ where the tail sums to n minus the head. Thus $a_n = a_{n-1} + a_{n-2}$ where a_{n-1} accounts for all sums where the head is 1 and a_{n-2} accounts for all sums where the head is 2.

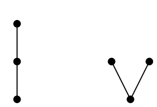
Thus a_n satisfies the Fibonacci recurrence and since $a_1 = F_2 = 1$ and $a_2 = F_3 = 2$, we have $a_n = F_{n+1}$ by induction.

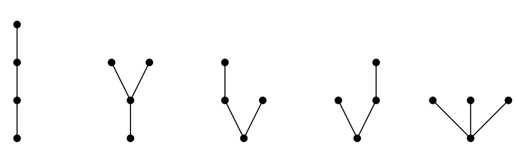
9. A *rooted plane tree* is a tree that has one root node planted in the ground and several subtrees branching off of it.

The word "plane" means that the order that the branches of the trees are in matter. So, for instance,  are all considered distinct.

These are the complete lists for $n = 1, 2, 3$

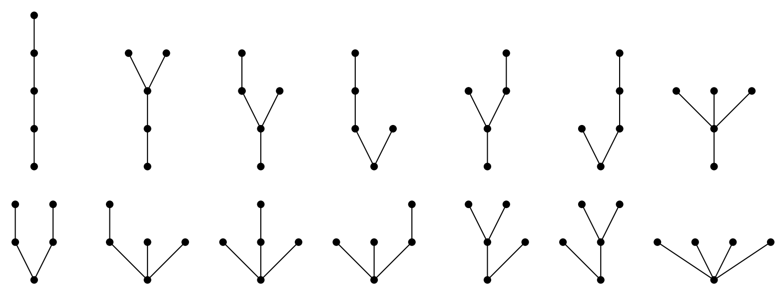
$n = 1$

$n = 2$ 

$n = 3$ 

Find the number of rooted plane trees with 4 edges.

Solution: 14.



10. Let N be the number of ordered pairs of integers (a, b) such that $a^2 + b^2 \leq 10^6$. Which of the following numbers is closest to N?

Solution: 3,150,000.
The region $x^2 + y^2 \leq 10^6$ in the xy-plane is a disk whose radius is $10^3 = 1000$, and thus, has an area of $\pi \cdot 10^6 \approx 3.14 \times 10^6$. Intuitively, since the disk is very large and nicely shaped, the number of lattice points $(x, y) = (a, b)$ inside the disk will be approximately its area, i.e. $N \approx 3.14 \times 10^6$, which suggests D is the closest answer choice. Although it is not necessary for competitors to do this on the multiple choice exam, we can guarantee that D is the closest answer choice by finding upper and lower bounds on N as follows.

Let $\mathcal{A}$ be the region of points (x, y) in the xy-plane which satisfy $\text{round}(x)^2 + \text{round}(y)^2 \leq 10^6$, where round$(x)$ rounds x to the nearest integer. This region is the union of N unit-squares which are centered at each of the lattice points (a, b) such that $a^2 + b^2 \leq 10^6$. Hence, the area of $\mathcal{A}$ is N. It is not hard to show that $\mathcal{A}$ fully contains the disk $x^2 + y^2 \leq 999^2$ and that $\mathcal{A}$ is fully contained inside the disk $x^2 + y^2 \leq 1001^2$. Hence,

$$\pi \cdot 999^2 \leq \text{Area}(\mathcal{A}) \leq \pi \cdot 1001^2.$$

Since $3.141 \leq \pi \leq 3.142$, we have $3,141,000 \leq \pi \cdot 10^6 \leq 3,142,000$. Also, it's not hard to show $1999\pi \leq 2001\pi \leq 7000$. Hence,

$$N = \text{Area}(\mathcal{A}) \geq \pi \cdot (1000-1)^2 = \pi \cdot 10^6 - 2000\pi + \pi \geq 3,141,000 - 7000 = 3,134,000$$

and

$$N = \text{Area}(\mathcal{A}) \leq \pi \cdot (1000+1)^2 = \pi \cdot 10^6 + 2000\pi + \pi \leq 3,141,000 + 7000 = 3,148,000.$$

From these two bounds, we see that D: 3,150,000 is the closest number to N.

11. The Tower of Hanoi game consists of three rods and a number of disks stacked on the first rod of different sizes with the largest disk on the bottom and decreasing in size towards the top. The objective is to move the stack from the first rod to a different rod one disk at a time and without placing a larger disk on top of a smaller disk.

    How many moves does it take to move a stack of 7 disks from one rod to the next?

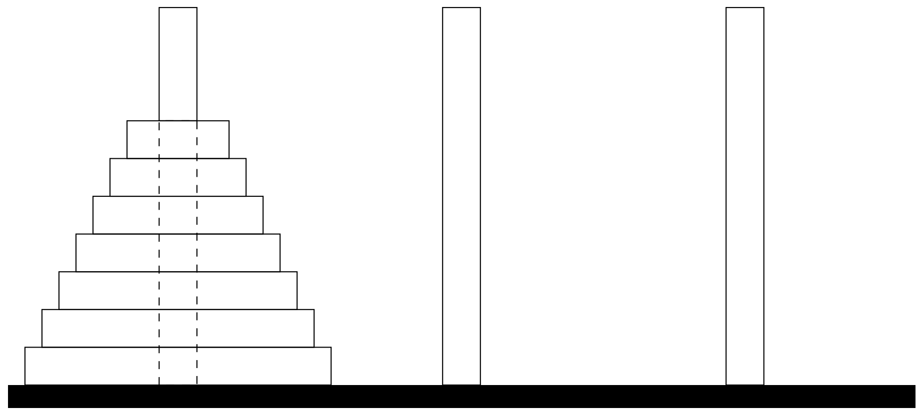


Figure 1: Tower of Hanoi Game

> **Solution: 127**.
> Let $h(n)$ be the number of moves required to move n disks from one peg to another. Then $h(1) = 1$ and $h(n+1) = h(n)+1+h(n) = 2h(n)+1$ (move n disks to a tertiary peg, then move 1 disk to the final peg, then move n disks from the tertiary peg to the final peg).
>
> So $h(7) = 2h(6) + 1 = 4h(5) + 3 = 8h(4) + 7 = 16h(3) + 15 = 32h(2) + 31 = 64h(1) + 63 = 127$.

12. Find $1^2 + 2^2 + 3^2 + \cdots + 100^2$.

    Note: you may use the identity $\sum_{k=1}^{n} \binom{k}{2} = \binom{n+1}{3}$.

> **Solution: 338350**.
> We use the Hockey-Stick Identity:
>
> $$\sum_{n=1}^{N} \binom{n}{k} = \binom{N+1}{k+1}.$$
>
> Since $n^2 = 2\binom{n}{2} + \binom{n}{1}$, we have
>
> $$\sum_{n=1}^{100} n^2 = 2\binom{101}{3} + \binom{101}{2}.$$

This gives

$$\frac{2(101)(100)(99)}{6} + \frac{(101)(100)}{2} = 101 \cdot 100 \cdot 33 + 101 \cdot 50 = 333300 + 5050 = 338350.$$

13. Call a set $S \subseteq \mathbb{R}^2$ *shattered* if for any subset $T \subseteq S$, there exists a disk $D \subseteq \mathbb{R}^2$ such that $D \cap S = T$. Find the largest natural number n for which there exists a shattered set of size n.

Solution: 3.
It can be shown by example that there exists a set of size 3 which is shattered.

To show that no set of size 4 is shattered, we suppose to the contrary that some set $\{x_1, x_2, x_3, x_4\} \subseteq \mathbb{R}^2$ is shattered. As disks are convex, the four points $x_1, \ldots, x_4$ must form a convex quadrilateral.

We may assume that the quadrilateral is oriented according to the order (x_1, x_2, x_3, x_4). Then by assumption, there exist disks D_1, D_2 such that $D_1 \cap \{x_1, \ldots, x_4\} = \{x_1, x_3\}$ and $D_2 \cap \{x_1, \ldots, x_4\} = \{x_2, x_4\}$. Remark that D_1 must include the line segment $\overline{x_1 x_3}$, and so $D_2 \setminus D_1$ is composed of two (disjoint) regions. This, however, is impossible, as the set difference of two disks is always connected.

14. Find the smallest integer $k \geq 1$ such that $403^k \equiv 1 \pmod{2020}$.

Solution: 4.
The prime factorization of 2020 is $2^2 \cdot 5 \cdot 101$. Letting $o(\cdot)_m$ denote the multiplicative order modulo m, $o(403)_{2020} = \text{lcm}\{o(403)_4, o(403)_5, o(403)_{101}\} = 4$.

15. Find $\sin(\pi/5)$.

Solution: $\sqrt{\frac{5}{8} - \frac{\sqrt{5}}{8}}$.
Let $\theta = \frac{\pi}{5}$. Since $2\theta = \pi - 3\theta$ and $\sin(\pi - A) = \sin(A)$, we have

$$\sin(2\theta) = \sin(3\theta).$$

Expanding this using $\sin(A+B) = \sin A\cos B + \sin B\cos A$, we obtain

$$\begin{aligned} 2\sin\theta\cos\theta &= 2\sin\theta\cos^2\theta + \sin\theta\cos 2\theta \\ &= 2\sin\theta(1-\sin^2\theta) + \sin\theta(1-2\sin^2\theta) \\ &= 3\sin\theta - 4\sin^3\theta \\ 2\cos\theta &= 3 - 4\sin^2\theta \\ 2\cos\theta &= -1 - 4\cos^2\theta. \end{aligned}$$

So by the quadratic formula we have

$$\cos\theta = \frac{2\pm\sqrt{20}}{8} = \frac{1+\sqrt{5}}{4}.$$

Note that $\cos\theta > 0$ so the other solution is extraneous.

Finally,

$$\sin\theta = \sqrt{1-\left(\frac{1+\sqrt{5}}{4}\right)^2} = \sqrt{\frac{16-6-2\sqrt{5}}{16}} = \sqrt{\frac{5}{8}-\frac{\sqrt{5}}{8}}.$$

16. How many 0's are there at the end of the integer $\frac{2021!}{1010!}$ in base 10?

Solution: 252.
It is an easy fact that this depends only on the power of 5 appearing in the prime factorization since there are many more even numbers (powers of 2) appearing. When p is a prime (such as $p = 5$), we define $v_p(n) = k$ if we can write $n = p^k m$ with $p \nmid m$. It is well-known, and easy to show, that

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

So we can therefore calculate

$$v_5(2021!) = 404 + 80 + 16 + 3, \text{ and } v_5(1010!) = 202 + 40 + 8 + 1.$$

The number we are after is

$$v_5\left(\frac{2021!}{1010!}\right) = v_5(2021!) - v_5(1010!) = 503 - 251 = 252.$$

17. Simplify $\sum_{k=0}^{1010} \binom{2020}{k}(-1)^k$. (Write it as a single expression with no summation.)

Note: the upper limit of the summation is 1010 not 2020.

Solution: $\mathbf{\frac{1}{2}\binom{2020}{1010}}$.

Note that

$$\begin{aligned} 0 = (1-1)^{2020} &= \sum_{k=0}^{2020} \binom{2020}{k}(-1)^k \\ &= \sum_{k=0}^{1009} \binom{2020}{k}(-1)^k + \binom{2020}{1010} + \sum_{k=1011}^{2020} \binom{2020}{k}(-1)^k \\ &= 2\sum_{k=0}^{1009} \binom{2020}{k}(-1)^k + \binom{2020}{1010} \end{aligned}$$

So

$$2\left(\binom{2020}{1010} + \sum_{k=0}^{1009} \binom{2020}{k}(-1)^k\right) = \binom{2020}{1010}.$$

Trial-and-error with smaller inputs should get you to the right answer.

18. How many functions f from $\{1,2,3,4,5\}$ to $\{1,2,3,4,5\}$ are there which are at most two-to-one? Meaning there are no three elements a, b, c such that $f(a) = f(b) = f(c)$.

Solution: 2220.

Let $n^{\underline{k}} = n(n-1)\cdots(n-k+1) = \binom{n}{k}k!$. This is the number of ways to pick k elements from n where the order they are picked in matters.

There are 5! permutations. There are $\binom{5}{2}5^{\underline{4}}$ ways for just one pair of elements to map to the same element. There are $\frac{1}{2}\binom{5}{1}\binom{4}{2}5^{\underline{3}}$ ways for two pairs of elements to map to the same element.

Therefore we get

$$5! + \binom{5}{2}5^{\underline{4}} + \frac{1}{2}\binom{5}{2}\binom{3}{2}5^{\underline{3}} = 2220.$$

19. For how many real numbers $1 \le x \le 2021$ is $\sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}$ an integer?

Solution: 44. We first find a closed form expression for $f(x) := \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}$ for $x>0$. Starting with $f(x)=\sqrt{x+f(x)}$, we can square both sides and move everything to the left side to obtain $f(x)^2 - f(x) - x = 0$. Then, using the quadratic formula gives us $f(x) = \dfrac{1 \pm \sqrt{1+4x}}{2}$. Since $f(x)>0$ for all $x>0$, we can discard the negative option to get $f(x) = \dfrac{1+\sqrt{1+4x}}{2}$. Since $f(x)$ is continuous, strictly increasing, and satisfies $f(1) = \dfrac{1+\sqrt{5}}{2} \in (1,2)$ and $f(2021) = \dfrac{1+\sqrt{8085}}{2} \in (45,46)$, we have that $f(x) = n$ has a unique solution $1 \le x \le 2021$ for each integer $n = 2,\ldots,45$. Hence, there are $45-1 = 44$ real numbers $1 \le x \le 2021$ for which $\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}$ is an integer.

20. What is the largest $n \le 500$ such that none of the coefficients of $(1+x)^n$ are divisible by 3?

Solution: 485.
According to Lucas's Theorem, if $n = \sum a_i 3^i$ is the base 3 expansion of n and $m = \sum b_i 3^i$ is the base 3 expansion of m, then

$$\binom{n}{m} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots \pmod 3.$$

For all of these terms to be non-zero, we need all of the a_i's to be either 1 or 2. So the maximum we can get below 500 is $3^5+2\cdot3^4+2\cdot3^3+2\cdot3^2+2\cdot3+2\cdot1 = 3^5+(3^5-1) = 485$.

21. There is a unique, strictly increasing function f from the positive integers to the positive integers satisfying $f(f(n)) = 3n$. Find $f(2021)$. You may use that $2021_{10} = 2202212_3$ in base 3.

Give your answer in base 3 but without any subscript. E.g. if the answer is "2021" you should input "2202212."

Solution: 12022120.
Let's start by finding the value of $f(1)$. This is positive so we have $1 \le f(1)$. Then, since f is increasing, we have $f(1) \le f(f(1)) = 3$. It won't work to have $f(1) = 1$ since then $f(f(1)) = 1$. Nor will it do for $f(1) \ge 3$ since then $f(f(1)) = 3 \ge f(3)$ and it is only possible for $3 \ge f(3)$ in a strictly increasing function if $f(1) = 1, f(2) = 2, f(3) = 3$ which we know doesn't work. Therefore, $f(1) = 2$. It follows next that $f(2) = f(f(1)) = 3$ and $f(3) = f(f(2)) = 6$.

We conclude that the first part of our table is as follows:

n	1	2	3
$f(n)$	2	3	6

We could continue, but let us pause to note that $f(3n) = f(f(f(n)) = 3f(n)$. So right away, we get that $f(3^n) = 3f(3^{n-1}) = 3^2 f(3^{n-2}) = \cdots = 3^n f(1) = 3^n \cdot 2$. Let us fill in some more of the table using $f(6) = f(f(3)) = 9$ and $f(9) = 3^2 \cdot 2$.

n	1	2	3	4	5	6	7	8	9
$f(n)$	2	3	6			9			18

Now we know that $f(4) = 7$ and $f(5) = 8$ since f is increasing. That brings us here:

n	1	2	3	4	5	6	7	8	9
$f(n)$	2	3	6	7	8	9			18

and now we know that $f(7) = f(f(4)) = 12$ and $f(8) = f(f(5)) = 15$. That fills out the table up to 3^2.

n	1	2	3	4	5	6	7	8	9
$f(n)$	2	3	6	7	8	9	12	15	18

Now let's repeat the process up to 3^3 and see if we notice any patterns. We will start by putting in the values of $f(12) = f(f(7)), f(15) = f(f(8)), f(18) = f(f(9)), f(27) = 3^3 \cdot 2$ that we know.

n	9	10	11	12	13	14	15	16	17	18	$\cdots$	27
$f(n)$	18			21			24			27	$\cdots$	54

Now again we see that for $f(10), \dots, f(18)$ the numbers must simply increase by 1 since there is no other option if f is increasing.

n	9	10	11	12	13	14	15	16	17	18	$\cdots$	27
$f(n)$	18	19	20	21	22	23	24	25	26	27	$\cdots$	54

And having done that, we can put in values for $f(19) = f(f(10)), \cdots f(26) = f(f(17))$. It is exactly the same pattern we saw between 3^1 and 3^2. The numbers increase by 1 at a time and then they start increasing by steps of 3.

In equations, what we are seeing is that

- $f(3^n) = 3^n \cdot 2$

- $f(3^n + k) = f(3^n) + k$ for $k \leq 3^n$
- $f(2 \cdot 3^n + k) = f(2 \cdot 3^n) + 3k$ for $k \leq 3^n$

And, plugging in what we know already about $f(3^n)$ and $f(2 \cdot 3^n) = 3^n f(2) = 3^{n+1}$, we see that

- $f(3^n + k) = 3^n \cdot 2 + k$ for $k \leq 3^n$
- $f(2 \cdot 3^n + k) = 3^{n+1} + 3k$ for $k \leq 3^n$

So finally, working in base 3, we have

$$\begin{aligned} f(2021_{10}) &= f(2202212_3) \\ &= f(2 \cdot 3^6 + 202212_3) \\ &= 3^7 + 3 \cdot 202212_3 \\ &= 10000000_3 + 2022120_3 \\ &= 12022120_3 \end{aligned}$$

Strictly speaking, one should show that the function

$$f(n) = \begin{cases} 2 \cdot 3^n + w_3 & \text{if } n = 1w_3 \text{ in base 3} \\ 3^{n+1} + w0_3 & \text{if } n = 2w_3 \text{ in base 3} \end{cases}$$

satisfies the functional identity $f(f(n)) = 3n$ and is strictly increasing. This can be proven using induction. However, in order to get the answer, an "educated guess" is sufficient.