# High School Math Competition – 2021
# Ciphering Solutions

Georgia Tech School of Math

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1. What is the number of ways to write 10 as a sum of one or more 1's and 2's where the order of the summands matters (e.g. $1 + 1 + 2$ and $1 + 2 + 1$ count as two ways)?

> **Solution:** 89
>
> Let $a_n$ be the number of ways to write $n$ as a sum of one or more 1's and 2's where the order of the summands matters. By computing small values of $a_n$, it is easy to see that this is the Fibonacci sequence. Indeed, any $n$ can be written as $(n-1)+1$ or $(n-2)+2$, and so there are $a_{n_1}$ ways to write $n$ as a sum of one or more 1's and 2's where the last summand is a 1 and $a_{n-2}$ ways to do so where the last summand is a 2. Therefore $a_n = a_{n-1} + a_{n-2}$, where $a_0 = 1$ and $a_1 = 1$.
>
> See also Problem 8 from the multiple choice exam.

2. Write $\sqrt{9 - 2\sqrt{14}} = a\sqrt{b} - c\sqrt{d}$ where $a, b, c, d$ are positive integers and $b, d$ are square-free integers (i.e. the square roots cannot be reduced further).

   Give your answer as an ordered quadruple $(a, b, c, d)$ with no spaces. E.g. `(3,5,1,7)`.

> **Solution:** `(1,7,1,2)`
>
> Start by computing $(a\sqrt{b}-c\sqrt{d})^2 = (a^2b+c^2d)-2ac\sqrt{bd}$ and compare with $9-2\sqrt{14}$. So $9 = a^2b + c^2d$ and $2ac = 2$. Since these are positive integers, we need $a = c = 1$ and hence the equations reduce to $9 = b + d$ and $bd = 14$. $b$ and $d$ are 7 and 2 in some order. But $\sqrt{2} - \sqrt{7} < 0$ and cannot be a square root of anything, so it must be
>
> $$\sqrt{7} - \sqrt{2}.$$

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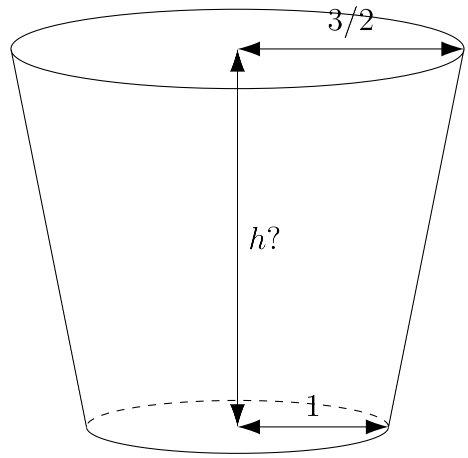
3. A glass has a radius of $3/2$ units at the mouth and a radius of 1 unit at the base with a conical middle. How tall must this glass be if it holds $100\pi$ units$^3$ of water?




Enter your answer as a fraction in reduced terms with no spaces. E.g. `500/3`.

**Solution:** `1200/19`

We can extend the cup to a cone of radius $3/2$ and height $3h$. It follows that $100\pi = \pi(3/2)^2\frac{3h}{3} - \pi\frac{2h}{3} = \pi\frac{19}{12}h$. So, $h = \frac{1200}{19}$.

4. What is the $x$-coordinate of the point $(-5\sqrt{2}, 2\sqrt{2})$ rotated about the origin by $135°$ counter-clockwise?

**Solution:** 3

Let $\theta$ be the angle between $(1,0)$ and $(-5\sqrt{2}, 2\sqrt{2})$. By the angle-addition formula,

$$\begin{aligned}
\cos(135+\theta) &= \cos(135)\cos(\theta) - \sin(135)\sin(\theta) \\
&= -\frac{\sqrt{2}}{2}\left[\cos\theta + \sin\theta\right] \\
&= -\frac{\sqrt{2}}{2}\Big[-\cos\big(\arctan(2/5)\big) + \sin\big(\arctan(2/5)\big)\Big] \\
&= -\frac{\sqrt{2}}{2}\Big[-\frac{5}{\sqrt{29}} + \frac{2}{\sqrt{29}}\Big] \\
&= \frac{3}{\sqrt{58}}.
\end{aligned}$$

Since the magnitude of $(-5\sqrt{2}, 2\sqrt{2})$ is $\sqrt{58}$, the answer is $\sqrt{58}\cdot\frac{3}{\sqrt{58}} = 3$.

5. What is the minimum value of the function $f(x) = -3x + 3 + 2x^2$ from $x = 0$ to $x = 1$? Enter your answer as a fraction in reduced terms with no spaces. E.g. `7/2`.

> **Solution:** `15/8`
>
> By completing the square, we obtain
>
> $$f(x) = 2x^2 - 3x + 3 = 2(x^2 - \tfrac{3}{2}x) + 3 = 2(x^2 - \tfrac{3}{2}x + \tfrac{9}{16}) - \tfrac{9}{8} + 3 = 2(x - \tfrac{3}{4})^2 + \tfrac{15}{8}.$$
>
> Since $(x - \frac{3}{4})^2 \geq 0$, we have $f(x) = 2(x - \frac{3}{4})^2 + \frac{15}{8} \geq \frac{15}{8}$ for all $x$ between 0 and 1. Furthermore, this lower bound can be attained, i.e., $f(\frac{3}{4}) = \frac{15}{8}$. Therefore, the minimum of $f(x)$ between $x = 0$ and $x = 1$ is $\frac{15}{8}$.

6. Let $f(x)$ be the fourth degree polynomial with $f(0) = 7$, $f(1) = 1$, $f(2) = 3$, $f(3) = 1$, $f(4) = 7$. What is $f(5)$?

> **Solution:** 57
>
> Since we are given the values of the fourth degree polynomial at five consecutive integers, the easiest way to calculate the value of the polynomial at the next integer is with a finite difference table. We begin by writing the sequence $f(0), f(1), f(2), \ldots$ in the top row. We form a sequence in each of the next four rows by computing the differences between consecutive elements of the sequence in the previous row:
>
> | | | | | | | | | | | | |
> |---|---|---|---|---|---|---|---|---|---|---|---|
> | $f(n)$ | 7 | | 1 | | 3 | | 1 | | 7 | | ? |
> | $\Delta f(n)$ | | -6 | | 2 | | -2 | | 6 | | ? | |
> | $\Delta^2 f(n)$ | | | 8 | | -4 | | 8 | | ? | | |
> | $\Delta^3 f(n)$ | | | | -12 | | 12 | | ? | | | |
> | $\Delta^4 f(n)$ | | | | | 24 | | ? | | | | |
>
> Since $f(n)$ is a quartic sequence, its first difference is a cubic sequence, its second difference is a quadratic sequence, its third difference is a linear sequence, and its fourth difference is a constant sequence. Hence, the sequence bottom row is $24, 24, 24, \ldots$. We can thus fill in the ? in the bottom row with 24, and then work upwards to fill in the ?s in the previous rows:
>
> | | | | | | | | | | | | |
> |---|---|---|---|---|---|---|---|---|---|---|---|
> | $f(n)$ | 7 | | 1 | | 3 | | 1 | | 7 | | 57 |
> | $\Delta f(n)$ | | -6 | | 2 | | -2 | | 6 | | 50 | |
> | $\Delta^2 f(n)$ | | | 8 | | -4 | | 8 | | 44 | | |
> | $\Delta^3 f(n)$ | | | | -12 | | 12 | | 36 | | | |
> | $\Delta^4 f(n)$ | | | | | 24 | | 24 | | | | |
>
> Therefore, $f(5) = 57$.

Alternate solution: We notice the five values of $f(x)$ provided uniquely determine the polynomial and these values have even symmetry about $x = 2$. Hence, $f(x)$ must have even symmetry about $x = 2$, which means it is of the form $f(x) = a(x-2)^4 + b(x-2)^2 + c$ for some constants $a$, $b$, and $c$. We just need to solve for the three coefficients. The condition $f(2) = 3$ becomes $c = 3$, $f(3) = 1$ becomes $a + b + c = 1$, and $f(4) = 7$ becomes $16a + 4b + c = 7$. Solving this system of equations yields $a = 1$, $b = -3$, and $c = 3$. Then, $f(x) = (x-2)^4 - 3(x-2)^2 + 3$, and thus, $f(5) = 3^4 - 3 \cdot 3^2 + 3 = 57$.

7. Find the largest number $n$ such that $n$ is prime and if you delete any subset of the digits of $n$, the result is still prime.

**Solution:** 73

Using the divisible-by-3 test where $n$ is divisible by 3 if and only if the sum of its digits are divisible by 3, we have the following rules:

- every digit of $n$ is prime
- $n$ can only contain 2 or 5 if it appears as the first digit.
- $n$ cannot contain more than one multiple of 3 because if we delete all the non-multiples of 3, the result will be a multiple of 3.
- $n$ cannot contain a digit which is congruent to 1 mod 3 and one that is congruent to 2 mod 3 at the same time.
- $n$ can contain at most 2 digits congruent to 1 or congruent to 2 mod 3.

So $n$ has at most 3 digits: it can have one multiple of three and two numbers which are both congruent to 1 or to 2 mod 3.

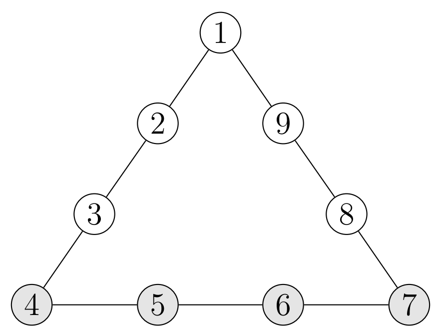
Only 2 and 5 are congruent to 2 (mod 3) and we can't have both 2 and 5 in our number. 7 is the only one-digit prime number congruent to 1 (mod 3). Since we can't repeat a digit, $n$ must have only 2-digits. By inspection, 73 is the largest 2-digit prime whose digits are prime.

8. Suppose you have 6 playing cards: two jacks, two queens, and two kings. All 6 cards are distinct, meaning that the two jacks are distinguishable, and similarly for the queens and kings. In how many ways can these cards be arranged in a line so that at least two cards of the same rank are adjacent?

**Solution:** 480

Let $A_j$ be the set of orderings in which the two jacks are adjacent, and similarly define $A_q$ and $A_k$. Then we are looking for $|A_j \cup A_q \cup A_k|$. Note that $|A_j| = 2 \cdot 5!$, $|A_j \cap A_q| = 2^2 \cdot 4!$, and $|A_j \cap A_q \cap A_k| = 2^3 \cdot 3!$. By inclusion-exclusion, the answer is $3(2 \cdot 5!) - 3(2^2 4!) + 2^3 3! = 480$.

9. An ant travels along the edges between the 9 nodes in the diagram below, starting from node 1. At each node, the ant flips a fair coin. If the coin is heads it moves clockwise, and it moves counter-clockwise otherwise. If it takes an average of $k$ jumps to reach one of the shaded nodes and an average of $m$ jumps to reach node 4, what is $m/k$?




**Solution:** 2

The first shaded node reached by the ant is, with equal probability, node 4 or node 7. If the ant reaches node 7 first, then it takes an average of $m$ additional jumps to get to node 4. So, $m = \frac{1}{2}k + \frac{1}{2}(k + m)$, so $m = 2k$.

10. Compute the area of the hexagon in the complex plane whose vertices are the solutions to $(z-2)^6 = -24\sqrt{3}$.

**Solution:** 9

The solutions $(z-2)^6 = -24\sqrt{3}$ form a regular hexagon whose vertices lie on the circle $|z-2| = (24\sqrt{3})^{1/6} = \sqrt{2\sqrt{3}}$. The hexagon can be broken up into 6 equilateral triangles whose side length is the radius of the circle, i.e., $r = \sqrt{2\sqrt{3}}$. Hence, its area is $6 \cdot \frac{r^2\sqrt{3}}{4} = 6 \cdot \frac{2\sqrt{3} \cdot \sqrt{3}}{4} = 9$.