# HSMC 2017 Free Response

1. What are the first three digits of the least common multiple of 234 and 360?

> **Solution:** 468. Note that $234 = 2 \cdot 3^2 \cdot 13$, and $360 = 2^3 \cdot 3^2 \cdot 5$. Thus, lcm $= 2^3 \cdot 3^2 \cdot 5 \cdot 13 = 10 \cdot 36 \cdot 13 = 4680$.

2. If Alice carries $G$ gallons of water in her backpack, she will ride her bicycle at a speed of $10 - G$ miles per hour. If Bob carries $G$ gallons of water in his backpack, it takes him $10 + G$ minutes to ride his bicycle one mile.
Alice and Bob divide 11 gallons of water between their backpacks and ride for 3 hours. It turns out that they both ride the same distance! What is that distance in miles?

> **Solution:** 12. Let $A$ be the amount of water Alice carries in gallons, and $11 - A$ the amount Bob carries. Because the distances are equal, we have the distance is $3(10 - A) = 3 \cdot \frac{60}{10+(11-A)} = \frac{180}{21-A}$. Rearranging gives $A^2 - 31A + 150 = 0$, the solutions are $A = 25, A = 6$. Clearly $A \leq 11$, so we have $A = 6$, and the distance is $3 \cdot (10 - 6) = 12$ miles.

3. Consider the minesweeper game on a $2 \times 50$ board. Suppose that initially all the squares on the bottom row are 1, indicating that there is one mine in their vicinity (here vicinity means squares which share a vertex or an edge, see figure), and there are no mines in the bottom row. The top row is all unknown. How many mines are there?

| | | | | |
|---|---|---|---|---|
| | $\bullet$ | $\bullet$ | $\bullet$ | |
| | $\bullet$ | $\star$ | $\bullet$ | |

The squares in the vicinity of the square with $\star$ are exactly the ones with $\bullet$

**Solution:** 17. Let us number the rows 1 and 2 for bottom and top, and columns 1,...,50 going from left to right. Now, we can describe each square by an ordered pair of natural numbers corresponding to row and column positions. Let us suppose there is a mine in the (2,1) square. Then there cannot be any mine on the squares (2,2) and (2,3), and so there has to be one on the square (2,4) since there is a 1 on the square (3,1). We see that this pattern (mine, empty,empty) has to repeat in the second row. In case there is no mine in the in the (2,1) square, there has to be one on the (2,2) square, and the same pattern has to repeat. Thus, the answer is $\lceil \frac{50}{3} \rceil = 17$.

4. The side lengths of a triangle are 280, 960, 1000. Find the radius of the circumcircle of the triangle.

**Solution:** 500. Note that $280^2 + 960^2 = 1000^2$, i.e. the given triangle is right angled. Hence circumradius is half the hypotenuse, i.e. 500.

5. The Fibonacci sequence is defined by the reccurence relation $F_1 = 1$, $F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$. The $n^{th}$ *Pisano period* is the period with which the Fibonacci numbers taken mod $n$ repeats. For example, the length of the $2^{nd}$ Pisano period is 3 because 110 is a full period of the Fibonacci numbers mod 2. Determine the length of $7^{th}$ Pisano period.

**Solution:** 16. The period is given by 1123516066542610.

6. There are 15 books in a row of a shelf. In how many ways can you pick 4 books (order of selection does not matter) so that no two are consecutive?

**Solution:** 495. There is a bijection between all such choices and sequences of 0's and 1's of length 15 with exactly 11 zeros and 4 ones, no two of which are consecutive. We write out 11 zeros and 12 stars (blank spaces) in between them, as follows:

$$*0*0*0*0*0*0*0*0*0*0*0*$$

Any choice of four stars corresponds to a choice of four books with desired property. Thus total number of choices is

$$\binom{12}{4} = \frac{12 \cdot 11 \cdot 10 \cdot 9}{24} = 495.$$

7. Suppose $a, b, c, d, e, f$ and $g$ are non-negative integers, not all zero, with greatest common divisor 1, and $9^a \cdot 10^b \cdot 11^c = 12^d \cdot 13^e \cdot 14^f \cdot 15^g$. Find $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2$.

**Solution:** 45. By prime factorization, we see that $c = e = f = 0$, and $b = g$, $2a = d + g$, $b = 2d$. Thus $2a = 3d$. We see that $\gcd(a, d)$ divides all given numbers, so $\gcd(a, d) = 1$ and hence $a = 3$, $d = 2$. Thus $b = g = 4$. Thus $a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 = 45$.

8. For the triangle $ABC$ we know that $AB = 42$, $AC = 36$ and $BC = 45$. $M$ is the midpoint of $BC$, and let $L$ be a point on $AB$ such that $CL$ is the internal angle bisector of $\angle BCA$, and $AM$ and $CL$ meet at point $P$. The ray $BP$ is intersecting $AC$ in point $Q$. Find $CQ$.

**Solution:** 20. We know $CM = BM$. Also $\frac{AL}{BL} = \frac{AC}{BC} = \frac{4}{5}$. By the Ceva's theorem $\frac{AL}{BL}\frac{BM}{CM}\frac{CQ}{QA} = 1$. Therefore $\frac{CQ}{QA} = \frac{5}{4}$ and $CQ = 20$.

9. A and B play a game: On A's turn, she rolls a six-sided die. A wins if the result is a 5. On B's turn, she flips a coin. B wins if the coin comes up heads. The players alternate turns until one wins, starting with A's turn. Let $\frac{m}{n}$ be the probability that A wins, expressed in lowest terms. What is $m + n$?

**Solution:** 9. Let $p$ the probability of A winning. With probability $\frac{1}{6}$ A wins on that turn, with probability $\frac{5}{6}$ B takes a turn. With probability $\frac{5}{12}$, B does not win, and the game resets to A's turn - in this case, she has probability $\frac{5}{12}p$ of winning. We find $p = \frac{1}{6} + \frac{5}{12}p$, solving shows $p = \frac{2}{7}$.

10. Suppose $a$ and $b$ are integers with $a + b - ab = 50$. How many possible solutions are there for the ordered pair $(a, b)$?

**Solution:** 6. Observe that $(1-a)(1-b) = 1 - 50 = -49$. Thus the only possibilities are $(1 - a) = \pm 1$ and $(1 - b) = \mp 49$;
or $(1 - a) = \pm 49$ and $(1 - b) = \mp 1$;
or $(1 - a) = \pm 7$ and $(1 - b) = \mp 7$
Thus there are six possible solutions $(0, 50), (2, -48), (50, 0), (-48, 2), (-6, 8)$ and $(8, -6)$.

11. An isosceles trapezoid $ABCD$ has an incircle. The bases of the trapezoid have lengths 36 and 64. Find the diameter of the incircle.

> **Solution:** 48. Let $AB = 64$ and $CD = 36$. If $CH$ is the height from $C$, $BCH$ is a right triangle with $BH = 14$. Now since the trapezoid has an incircle, $AB + CD = AD + BC$. Therefore, $AD = BC = 50$. We can find $CH = \sqrt{50^2 - 14^2} = 48$.

12. You have strawberry, vanilla and chocolate ice-creams, ten of each kind. In how many ways can you give six children one ice-cream each so that every type of ice-cream is given out?

> **Solution:** 540. The number of ways is the same as the number of surjective functions from $\{1, 2, 3, 4, 5, 6\}$ to $\{1, 2, 3\}$. By the inclusion exclusion principle this equals
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> $$3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 = 540.$$

13. Consider three concentric circles $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$, whose radii are in arithmetic progression with common difference 100 cm. Suppose you draw a tangent line (ray) from a point $P$ on the innermost circle $\mathcal{C}_1$, and it meets $\mathcal{C}_2$ and $\mathcal{C}_3$ at points $Q$ and $R$ respectively, with $PQ : QR = 2 : 1$. Find the radius (in cm) of the outermost circle $\mathcal{C}_3$.

> **Solution:** 550 (cm). Let $r_i$ be the radius of $\mathcal{C}_i$, for $i = 1, 2, 3$. Let $QR = a$ and $d = 100$ be the common difference. By Pythagorean theorem we have $r_2^2 = r_1^2 + 4a^2$, $r_3^2 = r_1^2 + 9a^2$. Using $r_1 = r_2 - d$ and $r_1 + r_3 = 2r_2$, we get, $d(2r_2 - d) = 4a^2$, $4dr_2 = 9a^2$ and thus $9d(2r_2 - d) = 16dr_2$, i.e. $9(2r_2 - d) = 16r_2$, and hence $r_2 = \frac{9}{2}d = 450$. Thus $r_3 = 550$.

14. You are in a game show where you are given 5 prizes and 5 distinct price tags to put on them. You win a prize if you correctly match it to its price. The problem is, you have no idea how much any of the items cost, so you choose randomly which tag to put on which prize. Let $p$ the probability that you win exactly 2 of the prizes. What is the closest integer to $1000p$?

> **Solution:** 167. There are $5! = 120$ possible ways to assign prices to prizes. There are $\binom{5}{2} = 10$ ways in which to assign 2 prices correctly; for each of these, there are 2 ways to assign the other 3 prices such that none of them is correct. Therefore the probability of getting exactly 2 prices correct is $\frac{10 \cdot 2}{120} = \frac{1}{6}$.

15. Suppose $x$ and $y$ are positive reals. Let $m$ be the minimum value $10x + 10y$ subject to the constraint $9x^2y^3 = 128$. What is the integer closest to $m$?

> **Solution:** 33. By the AM-GM inequality, we see that $x + y = x/2 + x/2 + y/3 + y/3 + y/3 \geq 5\sqrt[5]{(x/2)^2(y/3)^3} = \frac{10}{3}$ with equality if $x/2 = y/3$, so the minimum is attained at $m = \frac{100}{3}$.

16. In how many ways can you arrange 1 King, 1 Queen, 2 Bishops, 2 Knights and 2 Rooks in a row so that the King is in between Rooks, and there are an even number (including zero) of pieces between the 2 Bishops, and there are an even number of pieces between the 2 Knights?

> **Solution:** 576. The condition that there are an even number of pieces between the Bishops and Knights is the same thing as saying they are opposite colored, if we alternately color the squares of the row black and white. We pick places for bishops first in $4^2$ ways (4 choices for each color), knights in $3^2$ ways, pick places for queen in 4 ways, put Rook King Rook in that order in the remaining three squares. Total number of ways $= 16 \cdot 9 \cdot 4 = 576$.

17. Let $\frac{p}{q}$ be the fraction $\sum_{n=3}^{30} \frac{1}{n(n-1)(n-2)}$, expressed in lowest terms. What is $|p - q|$?

> **Solution:** 653. Note that
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> $$\frac{1}{n(n-1)(n-2)} = \frac{1}{2} \cdot \frac{n-(n-2)}{n(n-1)(n-2)} = \frac{1}{2} \cdot \left(\frac{1}{(n-1)(n-2)} - \frac{1}{n(n-1)}\right),$$
>
> We see that the given sum telescopes and we get
>
> $$\sum_{n=3}^{30} \frac{1}{n(n-1)(n-2)} = \frac{1}{2} \cdot \left(\frac{1}{2} - \frac{1}{870}\right) = \frac{217}{870}.$$

18. In 3-dimensional space, let $P$ be the plane $z = x + 3$ and $C$ be the cone $z^2 = 4(x^2 + y^2)$. Let $S$ be the intersection of $P$ and $C$. Let $A$ be the area of $P$ that is bounded by $S$. What is $(\frac{A}{\pi})^4$?

**Solution:** 576. It is well-known that when the intersection of a cone and plane bounds an area, the intersection $S$ is an ellipse. It is necessary to find the lengths $a$ and $b$ of the semi-axes, the area is $\pi ab$.
Because of symmetry across the $xz$-plane, the minor semi-axis is parallel to the $y$-axis, the major is in the $xz$-plane.
The length $a$ of the major semi-axis is $\frac{1}{2}$ the distance between the two points of intersection of $S$ with the $xz$-plane $y = 0$. These points solve the system $z = |2x|$, $z = x + 3$, the solutions are $x = 3, z = 6$ and $x = -1, z = 2$, so $a = \frac{1}{2}(4\sqrt{2}) = 2\sqrt{2}$.
The length $b$ of the minor semi-axis is the maximum value of $|y|$ in $S$. The equation relating $x$ and $y$ in $S$ is $4y^2 = -3x^2 + 6x + 9 = 12 - 3(x-1)^2$, the maximum value of $|y|$ is achieved at $x = 1$, $b = |y| = \sqrt{3}$.
Thus $A = 2\sqrt{6}\pi$ and so $(\frac{A}{\pi})^4 = 576$.

19. For a function $f$ from $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$ and a number $b_0$ in $\{1, 2, 3, 4, 5\}$, let $b_1 = f(b_0)$, $b_2 = f(b_1)$, and so on. How many choices of $b_0$ and $f$ are there such that $b_n$ is not 4 for any $n$ and $f$ is injective?

**Solution:** 240. There are $5 \cdot 5! = 600$ choices of a permutation $f$ and $b_0 \in [5]$. First note we need $b_0 \neq 4$, which leaves us with 480 choices. Set $b_0 \neq 4$, then select a random sequence by choosing $b_1, b_2, \cdots \in [5]$ uniformly without repeats (though $b_0$ can repeat). If we select $b_0$ before 4 in this random process, the sequence $b_n$ will never contain 4. On the other hand if we select 4 before $b_0$, then the sequence clearly includes 4. The probability of $b_0$ coming before 4 in a uniform random sequence is $\frac{1}{2}$, so there are $\frac{1}{2} \cdot 480$ ways.
**Alternate solution:** For any permutation, let us consider the different size of cycle that 4 is in, and use the addition principle to get total number of choices for $f$ and $b_0$.

| 4 is in a | choices for $f$ | choices for $b_0$ after $f$ is chosen |
|---|---|---|
| 5-cycle | 4! | 0 |
| 4-cycle | 4! | 1 |
| 3-cycle | $4 \times 3 \times 2!$ | 2 |
| 2-cycle | $4 \times 3!$ | 3 |
| 1-cycle | 4! | 4 |

Thus total number of choices $= 4! \cdot (1 + 2 + 3 + 4) = 240$.

20. Find the remainder when $415^{2017}$ is divided by 279.

**Solution:** 55. We see that $279 = 9 \cdot 31$; and $415 \equiv 1 \pmod 9$ and hence $415^{2017} \equiv 1 \pmod 9$. Also we have $415 \equiv 12 \pmod{31}$, and $2017 \equiv 7 \pmod{30}$ so by Fermat's Little Theorem we have $415^{2017} \equiv 12^7 \equiv 24 \pmod{31}$. Thus by Chinese Remainder Theorem we have that the remainder is 55.

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