3 Proof

1. Find all real numbers u so that the equation $(u+1)x^2 - (u^2-u)x + (u-2) = 0$ does not have two distinct real roots in x.

   Solution: When $u = -1$, the equation becomes $-2x-3 = 0$ and it has only one real root. Otherwise consider discriminant, by writing $w = (u+1)(u-2) = u^2-u-2$, we have $\Delta = (u^2-u)^2-4(u+1)(u-2) = (w+2)^2-4w = w^2+4 > 0$, so the discriminant is always positive and the equation would have two real roots. Therefore the only possible value of u is -1.

2. In a round robin tournament there are N players $A_1, \ldots, A_N$. Any two players play against each other exactly once, each win (resp. draw, loss) is awarded 2 points (resp. 1, 0 points). It turns out that after the tournament the scores of $A_1, \ldots, A_n$ are strictly decreasing in this order, but A_{i+1} defeated A_i for $i = 1, 2, \ldots, N-1$, and the match between A_1 and A_N ended in a draw. Find the minimum possible value of N.

   Solution: A_1 can have at most $2N-5$ points and A_N has at least 3 points, but the two players differ by at least $N-1$ points, so $2N-5 \geq 3+(N-1)$, or $N \geq 7$. $N = 7$ is possible: beside the given results, set the mactches between $A_2 - A_4, A_4 - A_6, A_2 - A_6, A_3 - A_5$ to be draws and in all other matches the smaller index player beats the bigger index one, it is easy to check the scores of $A_1, A_2, \ldots, A_N$ are $9, 8, \ldots, 3$ respectively.

3. Find all integers $n \geq 2$ such that the number 11111_n (in base n) is a perfect square.

   Solution: The number 11111_n is equal to $n^4+n^3+n^2+n+1$. Now notice that $(n^2)^2 = n^4 < n^4+n^3+n^2+n+1 < n^4+2n^3+n^2 = (n^2+n)^2$. Hence we must have $n^4+n^3+n^2+n+1 = (n^2+k)^2$ for some integer k between 1 and $n-1 < \sqrt{n^2}$ inclusive. If we divide $n^4+n^3+n^2+n+1$ by n^2, the quotient will be n^2+n+1, and the remainder will be $n+1$. If we divide $(n^2+k)^2 = n^4+2kn^2+k^2$ by n^2, the quotient will be n^2+2k and the remainder will be k^2. Hence, we have $n^2+n+1 = n^2+2k$, i.e. $n+1 = 2k$, and $n+1 = k^2$. Therefore $2k = k^2$, and so we must have $k = 2$. Indeed, when $n = 2k-1 = 3$, $11111_3 = 121_{10}$ is a perfect square.

4. An infinite sequence $a_1, a_2, a_3, \ldots$ of 1's and 2's is uniquely defined by the following properties:

   1. $a_1 = 1$ and $a_2 = 2$.

2. For every $n \geq 1$, the number of 1's between the n^{th} 2 and the $(n+1)^{\text{st}}$ 2 is equal to a_{n+1}.

Is the sequence periodic from the beginning?

Solution: No. Suppose for a contradiction that the sequence has period k, where k is chosen to be as small as possible. Consider first the case $a_k = 1$. Then the sequence can be grouped into repeated blocks as follows:

$$12112\ldots1\ 12112\ldots1\ 12112\ldots1\ldots$$

Hence the substring 112112 appears in the sequence. So we must have $a_i = a_{i+1} = 2$ for some i, but that is impossible as there are no 1's between the two 2's here.

Now consider the case $a_k = 2$. The sequence therefore looks like

$$12112\ldots2\ 12112\ldots2\ 12112\ldots2\ldots$$

By definition, the n^{th} 2 has exactly a_n 1's preceeding it, so replacing each 12 with a 1 and each 112 with a 2 yields back the original sequence. But the disjoint repeated blocks remain disjoint and identical after such replacement procedure. This gives a smaller period and a contradiction to the minimality of k.

5. Let $\triangle ABC$ be a triangle. Let D be a point on BC such that $AD \perp BC$. Let E and F be points on AB and AC respectively such that $DE \perp AB$ and $DF \perp AC$. Extend EF to meet the circumcircle of $\triangle ABC$ at X and Y respectively (X is on the arc AB and Y is on the arc AC). (a) Show that $AX = AD = AY$; (b) show that $\sin \angle A = \frac{XC}{AC} - \frac{XB}{AB}$. You can use the result of (a) for (b) even if you can not solve (a).

Solution: (a) Say XY meets AO at Z. We have $\angle AEZ = \angle ADF = \angle C$, the first equality follows from that A, E, D, F are concyclic; also $\angle ZAE = 90^\circ - \frac{1}{2}\angle AOB = 90^\circ - \angle C$, so $\angle AZE = 90^\circ$ and $AO \perp XY$, in particular $AX = AY$. Now $\angle AXE = \angle AYX = \angle ABX$, hence $\triangle AXE \sim \triangle ABX$; we also have $\triangle ADE \sim \triangle ABD$, thus $AX^2 = AE \cdot AB = AD^2$, that is $AX = AD$.

(b) By Ptolemy theorem and basic area formulae we have $XC \cdot AB - XB \cdot AC = XA \cdot BC = AD \cdot BC = 2S_{ABC} = AB \cdot AC \sin \angle A$, now divide both sides by $AB \cdot AC$.

Tiebreaker. Let S be a set on 2016 elements. Let $\star$ be a binary operation on S, that is, for any $a, b \in S$, we have that $a \star b$ is an element of S. Suppose that $(a \star b) \star c = a \star (b \star c)$. Suppose further that $a \star b = b \star a$ if and only if $a = b$. Show that for any $a, b, c \in S$, $(a \star b) \star c = a \star c$ for any a, b, c from S. (Partial credit will be given for any non-trivial progress toward this identity.)

Solution: First $(a \star a) \star a = a \star (a \star a)$ implies $a \star a = a$ for any $a \in S$; then $(a \star b \star a) \star a = a \star b \star a = a \star (a \star b \star a)$ implies $a \star b \star a = a$ for any $a, b \in S$; and finally $(a \star b \star c) \star (a \star c) = (a \star b \star c \star a) \star c = a \star c = a \star (c \star a \star b \star c) = (a \star c) \star (a \star b \star c)$ implies $a \star b \star c = a \star c$.

End of exam.