1. The towns $A$, $B$, and $C$ are connected by roads with at least one road between each town. The total number of ways to get from $A$ to $B$ without going through the same town twice is 11. The total number of ways to get from $A$ to $C$ without going through the same town twice is 14. What is the number of direct roads between $B$ and $C$?

**Solution:** Let the number of direct roads between $A$ and $B$ be $x$, between $A$ and $C$ be $y$, and between $B$ and $C$ be $z$. Then the total number of ways to get from $A$ to $C$ without going through the same town twice is $y + x \cdot z = 14$, and the total number of ways to get from $A$ to $B$ without going through the same town twice is $x + y \cdot z = 11$. Adding these two equations we get $(z+1)(x+y) = 25$. As $x, y, z \geq 1$, we must have $z + 1 = 5$, so $z = 4$.

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2. Let $\Delta ABC$ be a triangle with incenter $I$. Suppose the points diametrically opposite to $I$ in the circumcircles of $\Delta BCI$, $\Delta CAI$, and $\Delta ABI$ are $D, E$, and $F$ respectively. Show that $AD, BE$, and $CF$ intersect at a single point.
*Remark: The incenter of a triangle is the center of the circle inscribed in the triangle. It is a well known fact that the incenter is the point where the three bisectors of the triangle intersect.*

**Solution:** Let $O$ be the center of the circumcircle of $\Delta BCI$. Furthermore let $AI$ intersect the circumcircle of $\Delta ABC$ for the second time at the point $K$. By comparing angles we can see that $KB = KI = KC$. Thus we can conclude that $K \equiv O$. Therefore $A, I$, and $D$ are collinear. Analogously $B, I$, and $E$ and $C, I$, and $F$ are collinear. Thus the three lines intersect at $I$.

3. 2015 points are chosen in a $7 \times 7 \times 15$ rectangular prism. Prove that the distance between some two points is less than or equal to 1.

**Solution:** Assume to the contrary that no such pair exists. Then we have a collection of 2015 disjoint balls of radius $\frac{1}{2}$, centered at the given points, contained within the enlarged $8 \times 8 \times 16$ rectangular prism. The total volume of the balls is $2015 \cdot \frac{4}{3}\pi \left(\frac{1}{2}\right)^3 = 2015 \cdot \frac{\pi}{6}$, which is greater than the total available volume of 1024. This is a contradiction. So a pair of balls must overlap and their centers give the desired pair of points.

4. For each integer $n$, let $f(n)$ be the largest odd integer dividing $n$. For example, $f(3) = 3, f(8) = 1$, and $f(60) = 15$. Show that $f(n+1) + f(n+2) + \cdots + f(2n-1) + f(2n)$ is a perfect square for any positive integer $n$.

**Solution:** We claim that the $n$ numbers $f(n+1), \ldots, f(2n)$ are exactly the $n$ odd numbers $1, 3, \ldots, 2n-1$. If this was the case, we would have $f(n+1) + f(n+2) + \cdots + f(2n-1) + f(2n) = 1 + 3 + \ldots + (2n-1) = n^2$. To prove the claim notice that no two $f(i)$'s take the same value. Indeed suppose $f(i) = f(j) = k$. Then $i = k \times 2^a$ and $j = k \times 2^b$ for some $a \neq b$, say $a < b$. Then $\frac{j}{i} = 2^{b-a} \geq 2 > \frac{2n}{n+1}$, a contradiction to the fact that $i$ and $j$ are integers between $n+1$ and $2n$. So $f : \{n+1, n+2, \ldots, 2n\} \to \{1, 3, \ldots, 2n-1\}$ is an injection, which must be a bijection as the domain and range have the same number of elements.

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5. Show that it is possible to partition the set $\{1, 2, 3, \ldots\}$ into two (disjoint) sets $A$ and $B$ such that $A$ does not contain any non-trivial (difference not equal to 0) 3-term arithmetic progressions and $B$ does not contain any non-trivial infinite arithmetic progressions.
*Remark: An arithmetic progression is a sequence of numbers* $\{a_n\}_{n=1}^{\infty}$ *such that* $a_{n+1} - a_n = d$ *for some number* $d$ *called the difference of the progression.*

**Solution:** (Infinite) Arithmetic progressions in $\{1, 2, 3, \ldots\}$ are countable as each of them can be described as an ordered pair of positive integers $(a, d)$ where $a$ is the first term and $d$ is the difference. List these arithmetic progressions in certain order $P_1, P_2, P_3, \ldots$. We can construct a set $A$ by picking one number from each $P_i$ and making it an element of $A$. We want to do this in a way that ensures that $A$ does not contain any non-trivial 3-term arithmetic progressions. We proceed with the construction inductively. First we pick an arbitrary number $n_1$ from $P_1$. Now suppose we have picked numbers $n_i$ from $P_i$ for all $i = 1, 2, \ldots, k$ so that $n_1 < n_2 < \cdots < n_k$ and in such a way that there are no 3-term arithmetic progressions in the set $\{n_1, n_2, \ldots, n_k\}$. Now we pick a number $n_{k+1}$ from $P_{k+1}$ so that $n_{k+1} > 2n_k$. This is always possible as $P_{k+1}$ is unbounded. If we assume that $n_i < n_j < n_{k+1}$ is a 3-term arithmetic progression for some indexes $i < j$ we will have $2n_j = n_i + n_{k+1}$ which is impossible as $2n_j < n_{k+1}$ by construction. Thus the new set of numbers is increasing and does not contain any 3-term arithmetic progressions, as required. Now we can then take $B = \{1, 2, 3, \ldots\} \setminus A$. From our construction we can see that the set $B$ does not contain any non-trivial infinite arithmetic progressions. Thus the sets $A$ and $B$ constructed above give the required partition.

6. A triangulation of a polygon is a way to partition the polygon into triangles by adding diagonals that do not intersect each other (except possibly at the vertices of the polygon). Among all possible triangulations of a regular 2015-gon, what is the maximum possible number of acute-angled triangles one can get?

**Solution:** Think of the triangles as inscribed in a circle. Then a triangle is acute if and only if it contains the circumcenter in its interior. Indeed if it contains the circumcenter in its interior, all sides of the triangle correspond to arcs of length smaller than half the circumference of the circle, and therefore the associated angles are acute. Conversely, if the center lies outside of the interior of the triangle, one of its sides corresponds to an arc of length greater than or equal to half the circumference of the circle, and therefore the associated angle is not acute. Since 2015 is an odd number, there is no diagonal of the 2015-gon passing through the center. Thus, there is exactly one acute triangle in any triangulation.