1. Suppose that p_1 and q_1 are the roots of the quadratic $x^2 - a_1x + b_1$, and p_2 and q_2 are the roots of the quadratic $x^2 - a_2x + b_2$. Prove that $p_1 + p_2$ and $q_1 + q_2$ are the roots of $x^2 - (a_1 + a_2)x + b_1 + b_2$ if and only if p_1p_2 and q_1q_2 are the roots of $x^2 - a_1a_2x + b_1b_2$.

Solution: The fact that p_i and q_i are the roots of $x^2 - a_ix + b_i$ implies that $a_i = p_i + q_i$ and $b_i = p_iq_i$ for $i = 1, 2$.

The condition that $p_1 + p_2$ and $q_1 + q_2$ are the roots of $x^2 - (a_1 + a_2)x + b_1 + b_2$ is equivalent to $a_1 + a_2 = p_1 + p_2 + q_1 + q_2$ and $b_1 + b_2 = (p_1 + p_2)(q_1 + q_2)$. Writing all a_i and b_i in terms of p_i and q_i, this is

$$p_1 + p_2 + q_1 + q_2 = p_1 + p_2 + q_1 + q_2,$$

$$p_1q_1 + p_2q_2 = p_1q_1 + p_1q_2 + p_2q_1 + p_2q_2.$$

The first condition is always true so we can throw it out. The second can be simplified to $p_1q_2 + p_2q_1 = 0$.

Similarly the condition that p_1p_2 and q_1q_2 are the roots of $x^2 - a_1a_2x + b_1b_2$ is equivalent to $a_1a_2 = p_1p_2 + q_1q_2$ and $b_1b_2 = p_1p_2q_1q_2$. Or equivalently

$$p_1p_2 + p_1q_2 + q_1p_2 + q_1q_2 = p_1p_2 + q_1q_2,$$

$$p_1p_2q_1q_2 = p_1p_2q_1q_2.$$

Again the second condition can be thrown out, while the first simplifies to $p_1q_2 + p_2q_1 = 0$. Therefore both statements are equivalent.

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2. Let $x_1, x_2, \ldots, x_n$ be n real numbers such that $\sum_i x_i \geq 0$. What is the minimum number of nonempty subsets of the n numbers that have non-negative sums?

Solution: First, there will always be at least 2^{n-1} such subsets: the whole set is obviously one, and for any proper nonempty subset, either it or its complement has a nonnegative sum (as they are disjoint and sum up to a nonnegative number). Now consider $x_1 = n-1, x_2 = \ldots = x_n = -1$, no nonempty subsets not involving x_1 can have a nonnegative sum, while any subset containing x_1 has a nonnegative sum, so 2^{n-1} can be achieved.

3. Let ΔABC be an acute triangle, H, O be the orthocenter and circumcenter of ΔABC respectively. Obtain O' by reflecting O along the side BC, show that $O'H = R$, where R is the radius of the circumcircle of ΔABC. (Do not use any theorems about lengths involving O and H without proof.)

Solution: Extend BO to meet the circumcircle Γ of ΔABC at Z, we first show $OO' = CZ$: since BZ is a diameter of Γ, $\angle BCZ = \pi/2$ thus $CZ = 2OX = OO'$ as $OX//CZ$ and $BX = BC/2$ by properties of circumcenter. Next we show $OO' = AH$: both CH and ZA are perpendicular to AB by the definition of orthocenter and that BZ is a diameter, also that AH, CZ are perpendicular to BC, hence $AHCZ$ is a parallelogram and $CZ = AH$. Finally both AH and OO' are perpendicular to BC and $AH = OO'$, so $AOO'H$ is a parallelogram, and $O'H = OA = R$.

4. Find all ordered triples of prime numbers (p, q, r) such that $p^q + q^p = r$.

Solution: First, note that $r = p^q + q^p \geq 2^2 + 2^2 = 8$. Hence, $r \neq 2, 3$, and so r cannot be divisible by 2 or 3 (since r is prime). Now, if p, q are both even or both odd, then $p^q + q^p = r$ is even, a contradiction. Thus, exactly one of p, q is even, (WLOG, say p is odd and q is even), and so $q = 2$. Then we have $p^2 + 2^p = r$. Since p is odd, $2^p \equiv 2 \pmod 3$. If $p \neq 3$, then p is not divisible by 3, and so $p^2 \equiv 1 \pmod 3$. But then $r = p^2 + 2^p \equiv 1 + 2 \equiv 0 \pmod 3$, a contradiction. Thus, $p = 3$, which forces $r = 3^2 + 2^3 = 17$, which is prime. Hence, the only triples are $(p, q, r) = (2, 3, 17)$ or $(3, 2, 17)$.

5. Denote by A_n the number of ways to place $2n$ chips in an $n \times n$ table, at most one chip in each cell, so that every row and every column has exactly two chips. Show that there exists some integer N such that every A_n with $n \geq N$ is divisible by $2014! = 1 \times 2 \times 3 \times \ldots \times 2014$.

Solution: We say a way to place chips that satisfies the above condition as a "configuration". First we show that $A_n = \frac{n(n-1)}{2}(2A_{n-1} + (n-1)A_{n-2})$. There are $\frac{n(n-1)}{2}$ ways to place the two chips in the first row, say we place them to the first and second column and further assume the other chip in the first column is in row i. If the other chip C in row i is in the second column, then deleting the first row, row i together with the first two columns leaves a configuration of size $(n-2) \times (n-2)$ and such process can be easily reversed given the index of i, this account for the term $(n-1)A_{n-2}$. Otherwise move chip C to the second column and delete the first row and first column to get a configuration of size $(n-1) \times (n-1)$, and such process can be easily reversed given the choice of chip in the "second" column to be placed in the "first" column, this accounts the $2A_{n-1}$ term.

Now $A_{2\times 2014!} = 2014![(2\times 2014!-1)(2A_{2\times 2014!-1} + (2\times 2014!-1)A_{2\times 2014!-2})]$, $A_{2\times 2014!+1} = 2014![(2 \times 2014! + 1)(2A_{2\times 2014!} + (2 \times 2014!)A_{2\times 2014!-1})]$ are divisible by 2014!, and from the recurrent relation we know A_n is divisible by 2014! for $n \geq 2 \times 2014!$.

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6. (Tiebreaker) Let D be a function of the positive integers such that

- $D(1) = 0$,
- $D(p) = 1$ for every prime number p,
- $D(ab) = aD(b) + bD(a)$ for all integers a and b.

Determine all positive integers n such that $D(n) = n$, and show that no other positive integers satisfy this property.

Solution: The function D is sometimes called the *arithmetic derivative*, since it acts like a derivative on the set of integers.

First we generalize the product rule above to an arbitrary number of factors. If $n = a_1 \cdots a_k$ then $D(n) = \frac{n}{a_1}D(a_1) + \ldots + \frac{n}{a_k}D(a_k)$. We can prove this by induction on k. In particular, suppose that the rule holds for $k-1$ terms and consider n as the product of the terms, $a_1 \cdot a_2, a_3, \ldots, a_k$. Then

$$D(n) = \frac{n}{a_1a_2}D(a_1a_2) + \frac{n}{a_3}D(a_3) + \ldots + \frac{n}{a_k}D(a_k)$$

by the induction hypothesis. Substituting in $D(a_1a_2) = a_2D(a_1) + a_1D(a_2)$ we get the desired formula. Clearly the formula holds for $k = 2$ so we are done.

Now we apply this formula to the prime factorization of n. Let $n = p_1^{b_1} \cdots p_k^{b_k}$ where $p_1, \ldots, p_k$ are distinct primes. By grouping up all the terms for the same prime we get

$$D(n) = b_1\frac{n}{p_1}D(p_1) + \ldots + b_k\frac{n}{p_k}D(p_k) = \sum_{i=1}^{k} b_i\frac{n}{p_i} = n\sum_{i=1}^{k}\frac{b_i}{p_i}$$

since every $D(p_i) = 1$.

Given this formula, suppose we have that $D(n) = n$. Then

$$\sum_{i=1}^{k}\frac{b_i}{p_i} = 1.$$

If $k > 1$, then it is impossible for this to happen—if so, then $b_1/p_1 = 1 - \sum_{i=2}^{k} b_i/p_i = \frac{N}{p_2 \cdots p_k}$ for some integer N. For this to be true, $b_1 < p_1$, so $p_1 \nmid b_1$. But then $p_1N = b_1p_2 \cdots p_k$: the right-hand side of the equation is not divisible by p_1, a contradiction.

So $n = p^b$ for a single prime p and $b/p = 1$ so $b = p$. So the only solutions to the "differential equation" $D(n) = n$ are the integers $n = p^p$ for a prime p.