**2017 Georgia Tech High School Mathematics Competition**
**Proof Test**

**Instructions.** Congratulations on advancing to the proof portion of the competition! Do not open this envelope until instructed to do so. Please write your 5-digit ID number in the designated area at the bottom of this envelope.

This envelope contains the exam questions, answer sheets, and scratch paper. This exam consists of five questions and a tiebreaker question. You will have 120 minutes to complete as much of the exam as possible. Write your ID number and the problem number in the designated areas at the top of every answer sheet. Answer sheets without ID and question numbers will not be graded. If you need additional space, include a separate sheet of paper inside your problem folder, labeling it with your ID number and the problem number. Indicate page numbers as appropriate. When you are finished, place only the papers you wish to be graded in the envelope and give the envelope to a proctor.

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**2017 Georgia Tech High School Mathematics Competition**
**Proof Test**

1. Prove that for any positive real numbers $a, b, c, d$,

$$\frac{a^3+b^3+c^3}{a+b+c}+\frac{b^3+c^3+d^3}{b+c+d}+\frac{c^3+d^3+a^3}{c+d+a}+\frac{d^3+a^3+b^3}{d+a+b} \geq a^2+b^2+c^2+d^2.$$

**Solution:** Want to show:

$$\frac{a^3+b^3+c^3}{a+b+c} \geq 1/3(a^2+b^2+c^2)$$

$(a+b+c)\times(a^2+b^2+c^2) = a^3+b^3+c^3+ab^2+ba^2+ac^2+ca^2+bc^2+cb^2$ Since $a^3+a^3+b^3 \geq 3a^2b$ (AM-GM) similarly we can get corresponding inequality. So $ab^2+ba^2+ac^2+ca^2+bc^2+cb^2 \leq 1/3(2a^3+b^3+2a^3+c^3+ ... +2c^3+b^3) = 2a^3+2b^3+2c^3$, thus we get the desired inequality.

2. Let the inscribed circle in the triangle $\Delta ABC$ touch the side $AB$ at the point $D$. Let $I_1$ and $I_2$ be the centers of the inscribed circles in $\Delta ACD$ and $\Delta BDC$. Show that the circumscribed circle of $\Delta I_1I_2D$ touches the side $AB$ (only one point in common).

**Solution:** Let $P$ and $Q$ be the points where the inscribed circles of $ADC$ and $BDC$ touch $AD$ and $BD$ respectively. Then

$$DP = \frac{1}{2}(DC+DA-AC) = \frac{1}{2}(DC+DB-BC) = DQ.$$

Therefore the inscribed circles in $ADC$ and $BDC$ touch $CD$ in the same point. Now note that $\angle I_2I_1D = \angle DI_1P = \angle I_2DQ$.

3. For any $n$ and $k$, consider the following two sets of $k$-tuples $(T_1, \ldots, T_k)$ of subsets of $\{1, \ldots, n\}$:

   - The first set, where $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_k$.

- The second set, where $T_1, \dots, T_k$ are pairwise disjoint.

Find a bijection between these two sets and determine their size in terms of $n$ and $k$.

**Solution:** Given a tuple $(T_1, \dots, T_k)$ belonging to either set, define a function $f : \{1, \dots, n\} \to \{0, 1, \dots, k\}$ by

$$f(i) = \text{smallest } j \text{ such that } i \in T_j \text{ or } f(i) = 0 \text{ if } i \notin \bigcup_{j=1}^{k} T_j.$$

We can recover a $k$-tuple with $T_1, \dots, T_k$ pairwise disjoint by taking $T_j = f^{-1}(j)$ for $j = 1, \dots, k$. We can recover a $k$-tuple with $T_1 \subseteq \cdots \subseteq T_k$ by defining $T_j = f^{-1}(\{j, j+1, \dots, k\}$.

We therefore have a bijection between the two sets of tuples and the set of all functions $\{1, \dots, n\} \to \{0, 1, \dots, k\}$. Thus the sets have $(k+1)^n$ elements.

4. Prove that for any positive $k, d$ with $d \geq 2$, the sequence

$$(a_n) = \left( \binom{n-k}{k} \bmod d \right)$$

is always periodic.

**Solution:** Recall Pascal's Identity:

$$\binom{n-k}{k} = \binom{n-k-1}{k-1} + \binom{n-k-1}{k}.$$

This implies that the sequence $(a_n)$ depends only on the value of the vectors

$$v_r = \left( \binom{r}{0}, \binom{r}{1}, \dots, \binom{r}{k} \right) \bmod d$$

for $r < n - k$.

By the Pigeonhole Principle, the sequence $v_0, v_1, \dots$ eventually repeats since there are only $d^k$ possible vectors. It follows that the sequence $(a_n)$ is periodic.

5. Find all functions $f : \mathbb{N} \to \mathbb{N}$ for which the equality $f(f(m+n)) = f(m) + f(n)$ holds for all natural numbers $m$ and $n$.

**Solution:** If follows that for any $m$ and $n$,

$$f(f(f(f(m+n)))) = f(f(m)) + f(f(n)).$$

Therefore for all natural numbers $m$, $n$ and $p$ we have:

$$f(f(m)) + f(f(n+p)) = f(f(f(f(m+n+p)))) = f(f(m+n)) + f(f(p)).$$

Now since $f(f(m+n)) = f(m) + f(n)$ and $f(f(n+p)) = f(n) + f(p)$, we get:

$$f(f(m)) + f(n) + f(p) = f(f(p)) + f(m) + f(n).$$

Then for $p = 1$ we get $f(f(m)) = f(m) + c$ with $c = f(f(1)) - f(1)$. Therefore $f(m+n) = f(f(m+n)) - c = f(m) + f(n) - c$. Let $g = f - c$. so that $g(m+n) = g(m) + g(n)$. By induction it's easy to get that $g(n) = ng(1)$.Therefore $f = an + b$ where $a$ and $b$ are constants. Checking shows that $a = 1$ and $f(n) = n + b$ with $b$ - nonnegative integer.

6. (Tiebreaker) A *repunit* is a number of the form $111 \cdots 11$. Show that every positive integer whose base 10 expansion ends in a 1,3,7, or 9 divides infinitely many different repunits.

**Solution:** We can prove this using the pigeonhole principle. Let $n$ be any number ending in a 1, 3, 7, or 9. If we look at the first $n+1$ repunit numbers: $1, 11, 111, \ldots, \underbrace{11 \cdots 11}_{n+1 \text{ times}}$, then the pigeonhole principle tells us that if we take these numbers $\pmod{n}$, that at least two will be congruent. (This is because we have $n+1$ numbers and there are only $n$ remainders when dividing by $n$, so at least two numbers have to collide with the same remainder) If we take the difference of these two congruent repunits, then we get a number of the form $xy$ where $x$ is a repunit and $y = 10^d$ for some $d \in \mathbf{Z}$. Since this number is the difference of two numbers with the same remainder when divided by $n$, then $n \mid xy$. But we know that $(10, n) = 1$, which means $(y, n) = 1$, therefore $n \mid x$.

To show that $n$ divides infinitely many repunits, we take the first $m \cdot n + 1$ repunits. By the pigeonhole principle, there are at least $m$ repunits that have the same remainder when divided by $n$. If we take the difference of each pair of $m$ numbers, we will find $\binom{m}{2}$ unique repunits by the argument above. If we let $m$ go to infinity, then we will find infinite repunits divisible by $n$, since there are infinite repunits.

**End of exam.**

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