Georgia Tech High School Math Competition

Multiple Choice Test

March 10, 2018

- Each correct answer is worth one point; there is no deduction for incorrect answers.
- Make sure to enter your ID number on the answer sheet.
- You may use the test booklet as scratch paper, but no credit will be given for work in the booklet.
- You may keep the test booklet after the test has ended.

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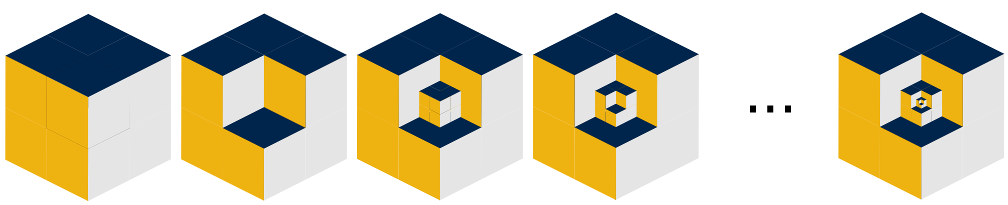
1. How many 4-letter strings can be formed from the letters in `MATHEMATICS`?
   - (a) 70
   - (b) 330
   - (c) 1680
   - (d) 2454
   - (e) 13440

Solution: D: 2454.
M=2, A=2, T=2, H=1, E=1, I=1, C=1, S=1. There are no ways to have all 4 letters the same. There are no ways to have 3 letters the same.
If there are 2 pairs of similar letters, there are $\binom{3}{2}$ ways to choose which pairs of letters to have, and these can be arranged in $\frac{4!}{2!2!}$ ways. $\binom{3}{2}\frac{4!}{2!2!} = 18$.
If there is 1 pair of similar letters, there are $\binom{3}{1}$ ways to choose the pair of letters and $\binom{7}{2}$ ways to choose the remaining letters. These can be arranged in $\frac{4!}{2!1!1!}$ ways. $\binom{3}{1}\binom{7}{2}\frac{4!}{2!1!1!} = 756$.
If all letters are different, there are $\binom{8}{4}$ ways to select these, and there are 4! ways to arrange these. $\binom{8}{4}4! = 1680$.
$18 + 756 + 1680 = 2454$.

2. Take a cube of length 1, and remove a cube of length 1/2 from one corner. In this removed corner, insert a cube of length 1/4 with a cube of length 1/8 removed from its corner. Iterate this process indefinitely.



What is the volume of this object?
   - (a) $\frac{7}{9}$
   - (b) $\frac{8}{9}$
   - (c) $\frac{15}{16}$
   - (d) $\frac{31}{32}$
   - (e) $\frac{8}{7}$

Solution: B: $\frac{8}{9}$.
At each step, we add $\left((\frac{1}{2})^i\right)^3$ volume and subtract $\left((\frac{1}{2})^{i+1}\right)^3$ volume to our object. This gives us the geometric series

$$\begin{aligned}\sum_{i=0}^{\infty}(-1)^i\left(\frac{1}{2}\right)^{3i} &= \sum_{i=0}^{\infty}(-1)^i\left(\frac{1}{8}\right)^i \\ &= \sum_{i=0}^{\infty}\left(-\frac{1}{8}\right)^i = \frac{1}{1-(-1/8)} \\ &= \frac{8}{9}.\end{aligned}$$

3. Consider a geometric progression with first term 2018 and common ratio 0.2. How many terms are greater than 1?

   (a) 1

   (b) 2

   (c) 3

   (d) 4

   (e) 5

Solution: E: 5 The fifth term is 3.2288 and the sixth is 0.64576. So 5.

4. Let N be some integer. Let N' be the integer obtained by permuting the digits of N in some way. If $N - N' = 314159?$, where the last digit of $N - N'$ is missing, then find the missing digit.

   (a) 0
   (b) 1
   (c) 4
   (d) 5
   (e) 8

Solution: C: 4.
It is a well-known fact that two numbers are congruent mod 9 if the sums of their digits are congruent mod 9. This can also be shown by expanding each number into its ones, tens, hundreds, etc, and comparing the residues mod 9.
Since N and N' have the same digits, we have $N \equiv N' \mod 9$, so $N - N' \equiv 0 \mod 9$. Thus $N - N'$ is divisible by 9, so the sum of its digits must be as well. Since $3 + 1 + 4 + 1 + 5 + 9 = 23$, it follows that the final digit must be 4.

5. How many ways are there to put 6 identical hats on 16 chairs (at most one hat per chair), arranged in a line, such that no two adjacent chairs have hats on them?

   (a) 330
   (b) 462
   (c) 792
   (d) 5040
   (e) 8008

Solution: B: 462.
Consider the sequences of gaps left after seating the 6 people. There are 7 gaps all together, say of lengths $n_1, \ldots, n_7$. Since the gaps between any two people has to be positive, we require $n_2, \ldots, n_6 > 0$. The sum of all the gaps, $n_1 + \cdots + n_7 = 10$.

Now let $n_1' = n_1 + 1$ and $n_7' = n_7 + 1$. Then $n_1', n_2, n_3, n_4, n_5, n_6, n_7' > 0$ and

$$n_1' + n_2 + n_3 + n_4 + n_5 + n_6 + n_7' = 12. \quad (1)$$

By the stars and bars method, the number of positive integer solutions to (**??**) is

$$\binom{12-1}{7-1} = 462.$$

6. $\cos\frac{\pi}{8} + \cos\frac{7\pi}{8} =$
   (a) $\sin\frac{\pi}{4}$
   (b) $\sin\frac{\pi}{2}$
   (c) $\cos\frac{\pi}{4}$
   (d) $\cos\frac{\pi}{2}$
   (e) None of the above.

Solution: D: $\cos\frac{\pi}{2}$. Indeed, the sum is actually 0.

7. Which of the following can be a value of n if $\binom{n}{k}/n$ is a positive integer for all $k = 1, 2, 3, \ldots, n-1$?
   (a) 14
   (b) 15
   (c) 16
   (d) 17
   (e) 18

Solution: D: 17. It's clear that if n is prime then $\binom{n}{k}$ is divisible by n for all choices of k. If n is even, then $nchoose2/n = (n-1)/2$ is not an integer because $n-1$ is odd. If n is a multiple of 3, then $nchoose3/n = (n-1)(n-2)/6$ is not an integer because neither $n-1$ nor $n-2$ is a multiple of 3. This rules out n = 14, 15, 16, and 18.

8. A point P is inside the square $ABCD$ with $AB = 8$. The distances between P and the points A, B and the side CD are all x. Find x.

   (a) 4

   (b) 5

   (c) 6

   (d) $4\sqrt{3}$

   (e) None of the above.

Solution: B: 5Let M be the center of the side AB. Then $AM = 4$, $PM = 8 - x$ and the triangle AMB is a right triangle. Then $16 + (8 - x)^2 = x^2$. We solve and get $x = 5$.

9. Let a_n and b_n be integers defined by the equation $a_n + b_n\sqrt{5} = (1 + \sqrt{5})^n$ for $n \geq 1$. Find a_6.

   (a) 576

   (b) 126

   (c) 256

   (d) 322

   (e) 610

Solution: A: 576.
Observe that

$$\begin{aligned} a_{n+1} + b_{n+1}\sqrt{5} &= (1 + \sqrt{5})(a_n + b_n\sqrt{5}) \\ &= (a_n + 5b_n) + (a_n + b_n)\sqrt{5}. \end{aligned}$$

Therefore we have the following recurrence relation with initial values $a_1 = 1, b_1 = 1$.

$$a_{n+1} = a_n + 5b_n; \quad b_{n+1} = a_n + b_n.$$

This gives

$$
\begin{aligned}
(a_1, b_1) &= (1, 1) \\
(a_2, b_2) &= (6, 2) \\
(a_3, b_3) &= (16, 8) \\
(a_4, b_4) &= (56, 24) \\
(a_5, b_5) &= (176, 80) \\
(a_6, b_6) &= (576, 256).
\end{aligned}
$$

Therefore $a_6 = 576$.

10. What is the number of integer solutions to the inequality:

$$\frac{x^2(x^2 + x - 12)}{(x^2 + 25)(-x^2 - x - 1)} > 0.$$

(a) 5
(b) 7
(c) 8
(d) 0
(e) Infinity.

Solution: A: 5 The solutions are given by $x \in (-4, 0) \cup (0, 3)$. So $-3, -2, -1, 1, 2$.

11. The inscribed circle in a right triangle splits the hypotenuse in two segments of lengths 4 and 6 respectively. Find the area of the triangle.

(a) 25
(b) 24
(c) 42
(d) 100
(e) None of the above.

Solution:

B: 24. We have that the hypotenuse has length 10 and the two sides have lengths $4+x$ and $6+x$ for some x. Solving for x yields $x = 2$ and then the area is $6 \cdot 8/2 = 24$.

12. Let $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}, n \geq 3$ be the Fibbonacci sequence. Find

$$\sum_{n=1}^{\infty} \frac{F_n}{12^n}.$$

(a) $\frac{1}{12}$

(b) $\frac{12}{131}$

(c) $\frac{12}{121}$

(d) $\frac{144}{131}$

(e) $\frac{144}{121}$

Solution: B: $\frac{12}{131}$.

Let

$$S = \sum_{n=1}^{\infty} \frac{F_n}{12^n}.$$

Note that

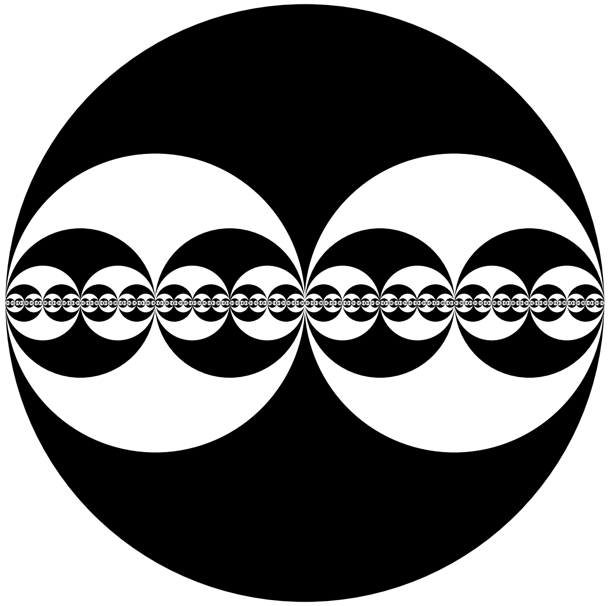
$$\begin{aligned} S &= \frac{1}{12} + \frac{1}{144} + \sum_{n=3}^{\infty} \frac{F_n}{12^n} \\ &= \frac{13}{144} + \sum_{n=3}^{\infty} \frac{F_{n-1} + F_{n-2}}{12^n} \\ &= \frac{13}{144} + \sum_{n=3}^{\infty} \frac{F_{n-1}}{12^n} + \sum_{n=3}^{\infty} \frac{F_{n-2}}{12^n} \\ &= \frac{13}{144} + \frac{1}{12} \sum_{n=2}^{\infty} \frac{F_n}{12^n} + \frac{1}{144} \sum_{n=1}^{\infty} \frac{F_n}{12^n} \\ &= \frac{13}{144} + \frac{1}{12}\left(S - \frac{1}{12}\right) + \frac{1}{144} S. \end{aligned}$$

Solving for S gives

$$S = \frac{13/144 - 1/144}{1 - 1/12 - 1/144} = \frac{12}{131}.$$

13. A dart is thrown at the figure below. Assuming that the dart is thrown uniformly in the region bounded by the outermost circle, what is the probability of hitting the black region?

(a) $\frac{1}{2}$



(b) $\frac{3}{5}$

(c) $\frac{2}{3}$

(d) $\frac{3}{4}$

(e) $\frac{5}{6}$

Solution: C: 2/3.

Solution 1: Let $A(r)$ be the area of the figure if the bounding circle had raidus r. Notice that the probability would then be the area of the figure divided by the area of the bounding circle hence would be $\frac{A(r)}{\pi r^2}$. Notice that the figure can be formed by taking a circle of radius r, deleting two circles of radius $r/2$, and then adding four copies of the original figure that are linearly scaled by a factor of $1/4$. This gives rise to the recurrence relation:

$$A(r) = \pi r^2 - 2\pi(r/2)^2 + 4A(r/4). \tag{2}$$

Note that in geometry, linearly scaling a two-dimensional figure by a factor of k will scale its area by a factor of k^2. Hence $A(r/4) = A(r)/16$. Substituting this in and simplifying equation (**??**) yields:

$$A(r) = \frac{\pi r^2}{2} + A(r)/4.$$

Solving for $A(r)$ yields,

$$A(r) = \frac{2}{3}\pi r^2$$

and hence the probability is

$$\frac{A(r)}{\pi r^2} = \frac{\frac{2}{3}\pi r^2}{\pi r^2} = \frac{2}{3}.$$

Solution 2: Let $A(r)$ be the area of the figure with bounding circle of radius r. Notice that the figure can be constucted by first taking a disk of radius r, deleting two disks of radius $r/2$, adding four disks of radius $r/4$, deleting eight disks of radius $r/8$, etc. This yields the following formula for $A(r)$:

$$\begin{aligned} A(r) &= \pi r^2 - 2\pi(r/2)^2 + 4\pi(r/4)^2 - 8\pi(r/8)^2 + \dots \\ &= \pi r^2 - \pi r^2/2 + \pi r^2/4 - \pi r^2/8 + \dots \\ &= \pi r^2(1 - 1/2 + 1/4 - 1/8 + \dots) \\ &= \pi r^2 \tfrac{1}{1-(-\frac{1}{2})} \\ &= \tfrac{2}{3}\pi r^2 \end{aligned}$$

here we used that for a geometric series, we have $1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$ when $|x| < 1$. In this case $x = \frac{-1}{2}$.

We see the probability is

$$\frac{A(r)}{\pi r^2} = \frac{\frac{2}{3}\pi r^2}{\pi r^2} = \frac{2}{3}.$$

14. Consider the following hexadecimal (base-16) number n below, where the standard digits for hexadecimal representations are used (so $A = 10, B = 11, C = 12, D = 13, E = 14, F = 15$):

$$n = F\underbrace{EE\ldots EE}_{2018}D\ DAD$$

where the 2018 signifies that there are 2018 E's in the hexadecimal representation above. What is the remainder when n is divided by 17?

(a) 1

(b) 3

(c) 4

(d) 9

(e) 10

Solution: A: 1
Since we are working in hexadecimal (base 16), one can observe n is congruent to the alternating sum of digits (from right to left) modulo 17. The proof of this is analogous to proving that (in base 10), any integer n is congruent to the alternating sum of its digits modulo 11. Knowing this, we see that:

$$\begin{aligned} n &\equiv D - A + (D - D) + (\underbrace{(E-E) + \cdots + (E-E)}_{1007 \text{ times}}) + F \pmod{17} \\ &\equiv D - A + 0 + (\underbrace{(0) + \cdots + (0)}_{1007 \text{ times}}) + F \pmod{17} \\ &\equiv D - A + F \pmod{17} \\ &\equiv 13 - 10 + 15 \equiv 18 \equiv 1 \pmod{17} \end{aligned}$$

15. For how many integers n between 1 and 2018 inclusive is n^n a perfect cube?

(a) 672

(b) 676

(c) 680

(d) 684

(e) 688

Solution: C: 680 If n is a multiple of 3 (i.e. $n = 3m$ for some integer m), then $n^n = n^{3m} = (n^m)^3$ is a perfect cube. If n is a perfect cube (i.e. $n = k^3$ for some integer k), then $n^n = (k^3)^n = (k^n)^3$ is a perfect cube. Now suppose n is not a multiple of 3 and is not a perfect cube. So there is some prime p which divides n exactly r times where r is not a multiple of 3. Then p divides n^n exactly rn times, and rn is not a multiple of 3. Thus, n^n is not a perfect cube. Therefore, n^n is a perfect cube if and only if n is a multiple of 3 or n is a perfect cube.

There are $\lfloor \frac{2018}{3} \rfloor = 672$ integers between 1 and 2018 which are multiples of 3. Since $12^3 = 1728 < 2018 < 2197 = 13^3$, there are 12 perfect cubes between 1 and 2018, of which 4 are also multiples of 3. Therefore, the number of integers n between 1 and 2018 which are multiples of 3 or perfect cubes is $672 + 12 - 4 = 680$.

16. The number $x = \sqrt[3]{5+\sqrt{17}} + \sqrt[3]{5-\sqrt{17}}$ is a root of which of the following polynomials?
    (a) $x^3 - 6x - 10$
    (b) $x^3 - 6x + 10$
    (c) $x^3 - 6x^2 - 10$
    (d) $x^3 - 6x^2 + 10$
    (e) $x^3 + 6x^2 + 6x + 10$

Solution: A: $x^3 - 6x - 10$. Let $a = \sqrt[3]{5+\sqrt{17}}$ and $b = \sqrt[3]{5-\sqrt{17}}$. Then since $x = a + b$, we have $x^3 = (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + b^3 + 3ab(a+b)$. Now, note that $a^3 + b^3 = (5+\sqrt{17}) + (5-\sqrt{17}) = 10$ and $ab = \sqrt[3]{5+\sqrt{17}}\sqrt[3]{5-\sqrt{17}} = \sqrt[3]{(5+\sqrt{17})(5-\sqrt{17})} = \sqrt[3]{5^2 - 17} = \sqrt[3]{8} = 2$. Hence, the equation $x^3 = a^3 + b^3 + 3ab(a+b)$ simplifies to $x^3 = 10 + 6x$. Thus, x is a root of $x^3 - 6x - 10$.

17. An integer x is picked uniformly at random from the set $\{0, 1, \ldots, 2018\}$. Consider the 10 consecutive integers: $x, x+1, \ldots, x+9$. Out of those ten, 5 are chosen at random. What is the probability that at least one of the chosen ones is divisible by 3?
    (a) $\frac{47}{52}$
    (b) $\frac{9}{13}$
    (c) $\frac{25}{29}$
    (d) $\frac{59}{63}$
    (e) 1

Solution: D: $\frac{59}{63}$. We have to consider two cases depending on x being divisible by 3. If yes, the probability is $p_1 = 1 - \binom{6}{5}/\binom{10}{5}$. If not, the probability is $p_2 = 1 - \binom{7}{5}/\binom{10}{5}$. The total probability is $1/3 \cdot p_1 + 2/3 \cdot p_2$ which evaluates to $\frac{59}{63}$.

18. A 3×4 rectangle has vertices $(0,0),(3,0),(3,4),(0,4)$. Point P, on the rectangle starts at $(1,1)$. The rectangle, along with point P, is rotated $90°$ clockwise about $(3,0)$, then $90°$ clockwise about $(7,0)$, then $90°$ clockwise about $(10,0)$, and finally $90°$ clockwise about $(14,0)$. After the four rotations, point P is now at $(15,1)$. Find the area bounded by the x-axis, $x=1$, $x=15$ and the path traced out by point P over the four rotations.

    (a) $6 + \frac{7}{2}\pi$

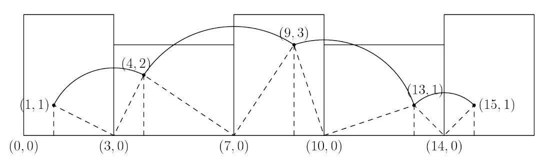
    (b) $12 + \frac{15}{2}\pi$

    (c) 10

    (d) $8 - \frac{3}{2}\pi$

    (e) $4 - \frac{5}{2}\pi$

Solution:



B: $12 + \frac{15}{2}\pi$ By drawing out the rotations, we can see that the desired area is the union of four quarter circles with radii $\sqrt{5}$, $\sqrt{13}$, $\sqrt{10}$, $\sqrt{2}$, and eight right triangles, with dimensions 1×2, 2×3, 3×1, 1×1 (with two of each). Thus, the total area is $\frac{1}{4}\pi(5+13+10+2) + 2 \cdot \frac{1}{2}(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 + 1 \cdot 1) = 12 + \frac{15}{2}\pi$.

19. How many functions $f : \{1,2,\ldots,8\} \to \{1,2,\ldots,8\}$ are there that satisfy the functional equation $f(f(x)) = x$?

    (a) 293

    (b) 435

    (c) 532

    (d) 764

    (e) 2343

Solution: D: 764.
Observe that such functions are permutations (i.e. bijections). For each $x \in \{1,\ldots,8\}$ either $f(x)=x$ or $f(x)=y$ and $f(y)=x$ with $x \neq y$. We can thus decompose f as a set of fixed points: $\mathrm{Fix}(f)=\{x : f(x)=x\}$ and a collection of pairs $\mathrm{Pairs}(f)=\{(x,y) : x<y, f(x)=y\}$. There is a bijection

$$f \longleftrightarrow (\mathrm{Fix}(f), \mathrm{Pairs}(f)).$$

To count the number of such functions, we first count the number of such functions with $\#\,\mathrm{Fix}(f)=k$. The number $8-k$ must be even since the number of pairs is $(8-k)/2$. If k, and hence $8-k$, is even, then there are $\binom{8}{k}$ ways to select the set $\mathrm{Fix}(f)$ and $(7-k)!!$ ways to pair up the numbers in $\{1,\ldots,8\}\setminus\mathrm{Fix}(f)$. Thus there are

$$\sum_{\substack{k=0\\ k \text{ even}}}^{8} \binom{8}{k}(7-k)!! = 764$$

such functions.

20. Call a permutation π of $\{1,2,\ldots,n\}$ an *alternating permutation of length n* if

$$\pi(1) < \pi(2) > \pi(3) < \pi(4) > \pi(5) < \cdots .$$

Find the number of alternating permutations of length 7.

(a) 272

(b) 273

(c) 274

(d) 275

(e) 276

Solution: A: 272.
Let $\mathcal{E}_n$ denote the set of alternating permutations of length n with n even and $\mathcal{O}_n$ denote the number of alternating permutations of length n with n odd. By removing $\pi(i)=n$ from the alternating sequence, we obtain two smaller alternating permutations:

$$\pi(1) < \pi(2) > \pi(3) < \cdots > \pi(i-1) \text{ and } \pi(i+1) < \pi(i+1) > \pi(i+2) < \cdots .$$

We also obtain a partition of $\{1, \ldots, n-1\}$ into two subsets:

$$(\{\pi(j) : j < i\}, \{\pi(j) : j > i\})$$

This decomposition gives us the following bijections:

$$\mathcal{E}_{n+1} \longleftrightarrow \bigcup_{i=0}^{n} \mathcal{B}(n,i) \times \mathcal{O}_i \times \mathcal{E}_{n-i},$$
$$\mathcal{O}_{n+1} \longleftrightarrow \bigcup_{i=0}^{n} \mathcal{B}(n,i) \times \mathcal{O}_i \times \mathcal{O}_{n-i},$$

where $\mathcal{B}(n,i)$ is the set of partitions of $\{1, \ldots, n\}$ into two sets of size i and size $n-i$ respectively. Observe that $\#\mathcal{B}(n,i) = \binom{n}{i}$.

Let $E_n = \#\mathcal{E}_n$ and $O_n = \#\mathcal{O}_n$. Since the above unions are disjoint, these bijections tell us that

$$E_{n+1} = \sum_{i=0}^{n} \binom{n}{i} O_i E_{n-i},$$
$$O_{n+1} = \sum_{i=0}^{n} \binom{n}{i} O_i O_{n-i}.$$

The initial values are $E_0 = 1, E_1 = 0, O_0 = 0, O_1 = 1$.

Performing the recurrence gives $O_7 = 272$.

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