Georgia Tech High School Math Competition

Multiple Choice Test

February 28, 2015

- Each correct answer is worth one point; there is no deduction for incorrect answers.
- Make sure to enter your ID number on the answer sheet.
- You may use the test booklet as scratch paper, but no credit will be given for work in the booklet.
- You may keep the test booklet after the test has ended.

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1. Which of the following numbers is between $\frac{22}{7}$ and $\frac{31}{10}$?

   (A) $\frac{53}{17}$

   (B) $\frac{61}{20}$

   (C) 3.15

   (D) All of the above

   (E) None of the above

Solution: A. By direct comparison.

2. A triangulation of a polygon is a way to partition the polygon into triangles by adding diagonals that do not intersect each other (except possibly at the vertices of the polygon). How many diagonals does one need to add in order to triangulate a regular 2015-gon?

   (A) 2012

   (B) 2013

   (C) 2014

   (D) 2015

   (E) 2016

Solution: A. In general, one needs $n-3$ diagonals to triangulate a regular n-gon. A rigorous way to see this is by first considering the sum of the interior angles. From this consideration one can conclude that there have to be exactly $n-2$ triangles in any triangulation. Then note that every diagonal added is shared by two triangles while every edge of the polygon is shared by one. We can conclude exactly $n-3$ diagonals are required.

3. Which of the following polynomials has only integer roots?

   (A) $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 1$

   (B) $2x^6 - 28x^3 + 2015$

   (C) $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x$

   (D) $2x^8 + 28x^4 + 2015$

   (E) $x^7 + 7x^6 + 21x^5 + 35x^4 + 35x^3 + 21x^2 + 7x + 2$

Solution: A. Observe that the coefficients are a row of Pascal's triangle. By the Binomial Theorem this is $(x+1)^7$.

4. Consider the binary number $A = 111111111_2$ and the ternary number $B = 200222_3$. What is the value of $A - B$?

   (A) -2

   (B) -1

   (C) 0

   (D) 1

   (E) 2

Solution: B. By direct calculation we find that the decimal representation of the ternary number is 512, while that of the binary number is 511.

5. We say a three digit number $\overline{abc}$ is "progressive" if a, b, and c form an arithmetic progression with non-zero difference, i.e., $a = b + k = c + 2k$ for some nonzero integer k. What is the sum of the largest and the smallest three digit "progressive" numbers?

   (A) 912

   (B) 999

   (C) 1110

   (D) 1197

   (E) 1308

Solution: C. The largest and smallest "progressive" numbers are 987 and 123 respectively.

6. I traveled to an island inhabited by knights who always tell the truth and knaves who always lie. Furthermore, the island's inhabitants only answer questions with "yes" or "no". Here, I met Alice and Bob. I asked Alice, "Are you both knights?" After Alice gave me an answer I figured out the type of each. What type was each?

   (A) Both were knights.

   (B) Both were knaves.

   (C) Alice was a knight and Bob was a knave.

   (D) Bob was a knight and Alice was a knave.

   (E) We cannot determine the types without knowing Alice's answer.

Solution: C. Alice would answer, "No," only if she was a knight and Bob was a knave. In any of the other three situations her answer would have been "Yes." As the speaker could figure out their types after hearing Alice's answer, she must have answered, "No."

7. Denote by $a_1 \vee a_2 \vee \ldots \vee a_n$ the maximum number in $\{a_1, a_2, \ldots, a_n\}$ and similarly denote by $a_1 \wedge a_2 \wedge \ldots \wedge a_n$ the minimum number in $\{a_1, a_2, \ldots, a_n\}$. Then what is $(1 \wedge 2 \wedge 3 \wedge \ldots \wedge 39 \wedge 40) \vee (1 \wedge 2 \wedge 3 \wedge \ldots \wedge 39 \wedge 41) \vee \ldots$ where the $\vee$ is taken over all terms of the form $a_1 \wedge a_2 \wedge \ldots \wedge a_{40}$ where $\{a_1, a_2, \ldots, a_{40}\}$ is a 40-element subset of $\{1, 2, , \ldots, 100\}$?

   (A) 39

   (B) 40

(C) 60

(D) 61

(E) 100

Solution: D. Every term $a_1 \wedge a_2 \wedge \ldots \wedge a_{40}$ is at most 61. This value is achieved by the term $61 \wedge 62 \wedge \ldots \wedge 100$.

8. What is the remainder of 2015^{2015} when divided by 15?
   (A) 0
   (B) 3
   (C) 5
   (D) 10
   (E) None of the above

Solution: C. 2015 gives remainder 5 when divided by 15. So 2015^2 gives remainder 10 when divided by 15, and 2015^3 gives remainder 5 when divided by 15. Thus it is easy to see that odd powers of 2015 give remainder 5 when divided by 15.

9. $AB = 2\sqrt{3}$ and $CD = 2$ are two chords on a circle with center O. It is known that the ratio of the distance from O to AB and the distance from O to CD is $1 : \sqrt{3}$. What is the radius of the circle?
   (A) 1
   (B) $\sqrt{2}$
   (C) 2
   (D) $2\sqrt{2}$
   (E) 3

Solution: C. Let the radius of the circle be r. Then the distances from O to AB and CD are $\sqrt{r^2 - \sqrt{3}^2}$ and $\sqrt{r^2 - 1^2}$ respectively. Then $\dfrac{\sqrt{r^2-3}}{\sqrt{r^2-1}} = \dfrac{1}{\sqrt{3}}$ gives $r = 2$.

10. Suppose $1 + \frac{1}{b} + \frac{1}{b^2} + \frac{1}{b^3} + \cdots = 2015$ and $\frac{1}{c} + \frac{1}{c^2} + \frac{1}{c^3} + \cdots = 2013$. What is the ratio $\frac{c}{b}$?
   (A) $\dfrac{2012 \times 2014}{2013 \times 2015}$
   (B) $\dfrac{2012 \times 2015}{2013 \times 2014}$
   (C) $\dfrac{2013 \times 2014}{2012 \times 2015}$
   (D) $\dfrac{2014^2 - 1}{2014^2}$

(E) $\dfrac{2014^2}{2014^2-1}$

Solution: E. The first equation implies $\dfrac{1}{b}=\dfrac{2014}{2015}$ and the second implies $\dfrac{1}{c}=\dfrac{2013}{2014}$, so the answer is $\dfrac{1/b}{1/c}=\dfrac{2014^2}{2013\times 2015}=\dfrac{2014^2}{2014^2-1}$.

11. In how many ways can one choose 5 cells from a 3×3 table such that three cells lie in the same row or column?

(A) 72

(B) 81

(C) 90

(D) 96

(E) 126

Solution: B. There are 6 ways to choose the row or column and $\binom{6}{2}=15$ ways to choose the position of the remaining two cells. However, in our counting above we have counted the $3\times 3=9$ cases in which we have chosen 3 cells on the same row and 3 cells on the same column twice. Thus we get a total of $6\cdot 15-9=81$ possible choices.

12. Consider the sum $A=1^2-2^2+3^2-\cdots-2014^2+2015^2$. What is the value of A?

(A) 2028098

(B) 2029105

(C) 2030112

(D) 2031120

(E) 2282015

Solution: D. We can observe that $(2k+1)^2-(2k)^2=4k+1$. Thus $A=1^2-2^2+3^2-\cdots-2014^2+2015^2=1+(4+1)+\cdots+(4\cdot 1007+1)=1008+4(1+2+\cdots+1007)=1008+2\cdot 1007\cdot 1008=1008\cdot 2015=2031120$.

13. Let n be a positive integer such that $\dfrac{n^3+24n+1959}{n-2}$ is also an integer. How many possible values of n are there?

(A) 2

(B) 8

(C) 9

(D) 16

(E) There are infinitely many possibilities for n

Solution: C. Notice that $\dfrac{n^3+24n+1959}{n-2} = n^2+2n+28+\dfrac{2015}{n-2}$, so the fraction is an integer if and only if $n-2$ is a factor of 2015. There are 8 positive factors of 2015, and together with the case $n=1$ there are 9 possible values of n.

14. Three chess players A, B, and C participate in a tournament. Players A and B play first; the loser then plays C. The winner of the match with C then plays the winner of the first match (A or B) in the final. Suppose the only possible outcome of a match is either winning or losing with equal probability for each player. What is the probability that A reaches the final and wins it?

(A) $\dfrac{1}{4}$

(B) $\dfrac{1}{3}$

(C) $\dfrac{3}{8}$

(D) $\dfrac{2}{5}$

(E) $\dfrac{1}{2}$

Solution: C. Note the winning chance for C is $\dfrac{1}{4}$ as he has to win both matches to win the title. By symmetry the winning chance of A and B separately is $\dfrac{1}{2}\left(1-\dfrac{1}{4}\right) = \dfrac{3}{8}$.

15. Three circles of radius r are arranged in the plane such that they are mutually tangent and no circle lies inside any of the others. Their centers all lie on a circle of radius 1. Compute r.

(A) $\frac{1}{2}$

(B) 1

(C) $\frac{\sqrt{3}}{2}$

(D) $\frac{\sqrt{2}}{2}$

(E) $\sqrt{3}$

Solution: C. Let O_1, O_2, and O_3 be the centers of the three circles of radius r, and let O be the center of the circle of radius 1. We have that $\angle O_1OO_2 = 120°$. Then by the Law of Cosines for triangle O_1OO_2, we find that $(2r)^2 = 3$. Thus $r = \frac{\sqrt{3}}{2}$.

16. If we role a fair die 3 times, what is the probability that the three numbers we get, in the order we get them, form a strictly increasing sequence?

(A) $\frac{20}{216}$

(B) $\frac{40}{216}$

(C) $\frac{60}{216}$

(D) $\frac{1}{6}$

(E) $\frac{1}{2}$

Solution: A. First, the probability of getting three different numbers is $\frac{6 \cdot 5 \cdot 4}{6^3} = \frac{20}{6^2}$. Exactly one of the 6 possible orders would give an increasing sequence. The total probability is $\frac{20}{6^3}$.

17. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two mutually tangent circles whose centers are distance 3 apart. Suppose the x-axis intersects $\mathcal{C}_1$ at the points $(3 - 2\sqrt{6}, 0)$ and $(3 + 2\sqrt{6}, 0)$, and it is tangent to $\mathcal{C}_2$ at the point $(3, 0)$. What is the radius of $\mathcal{C}_2$?

(A) 1

(B) 2

(C) 3

(D) 4

(E) 5

Solution: B. Denote by O_1 the center of $\mathcal{C}_1$, by O_2 the center of $\mathcal{C}_2$, by R the radius of $\mathcal{C}_1$, and by r the radius of $\mathcal{C}_2$. Furthermore let $P_1 = (3-\sqrt{6}, 0)$, $P_2 = (3+2\sqrt{6}, 0)$, and $Q = (3, 0)$. Finally, let O_1O_2 intersect $\mathcal{C}_1$ at the points X and Y. We can easily see that Q is the midpoint of the segment P_1P_2. Thus the points X, O_1, Q, O_2 and Y lie on the same line. Thus $P_1Q \cdot QP_2 = XQ \cdot QY$. From here we get $24 = P_1Q \cdot P_2Q = XQ \cdot QY = (2R-2r) \cdot 2r = 4r(R-r) = 12r$. Hence $r = 2$.

18. Let ΔABC be a triangle with area 1. Let D, E, and F be points on BC, CA, and AB, respectively, such that $BD = DC$, $CE = 2EA$, and $AF = 3FB$. What is the area of ΔDEF?

    (A) $\frac{1}{4}$

    (B) $\frac{5}{18}$

    (C) $\frac{7}{24}$

    (D) $\frac{5}{12}$

    (E) $\frac{1}{3}$

Solution: C. Using $\text{Area}(\Delta XYZ) = \frac{1}{2}XY \cdot YZ \sin(\angle XYZ)$, we see that $\text{Area}(\Delta AFE) = \frac{1}{4}$, $\text{Area}(\Delta FBD) = \frac{1}{8}$, and $\text{Area}(\Delta CDE) = \frac{1}{3}$. Thus, $\text{Area}(\Delta DEF) = \frac{7}{24}$.

19. If x_1 and x_2 are the roots of $x^2 - 28x + 2015$, what is the value of $(x_1 + 1)(x_2 + 1)$?

    (A) 28

    (B) 2015

    (C) 2042

    (D) 2043

    (E) 2044

Solution: E. $(x_1 + 1)(x_2 + 1) = x_1x_2 + x_1 + x_2 + 1 = 2015 + 28 + 1 = 2044$.

20. What is the highest power of 28 that divides (2015!)?

    (A) 199

    (B) 256

    (C) 287

    (D) 333

    (E) 356

Solution: D. This would be the same as the highest power of 7 dividing 2015!, which is given by $\left\lfloor \frac{2015}{7} \right\rfloor + \left\lfloor \frac{2015}{7^2} \right\rfloor + \left\lfloor \frac{2015}{7^3} \right\rfloor = 287 + 41 + 5 = 333$.

21. Let A, B, C, and D be four points on a circle with center O such that $OABC$ is a parallelogram and O lies inside the interior of $\triangle DAC$. Find $\angle OAD + \angle OCD$.

   (A) 45°

   (B) 60°

   (C) 75°

   (D) 90°

   (E) 120°

Solution: B. Since $OA = OB = OC = AB$, we know $\triangle ABO$ and $\triangle CBO$ are equilateral triangles. Hence, $\angle AOC = 120°$. Now, $\angle OAD + \angle OCD = \angle ODA + \angle ODC = \angle ADC = \frac{1}{2}\angle AOC = 60°$.

22. Joe frequently plays a game where he tosses a fair coin until he obtains two heads in a row. On average how many times will he toss the coin?

    (A) 3

    (B) 4

    (C) 5

    (D) 6

    (E) 7

Solution: D. Let E be the expected value of the number of times he will toss the coin. Furthermore let E_H be the expected value if the first outcome is H, and let E_T be the expected value if the first outcome is T. We have that $E_T = 1 + E$ and $E_H = \frac{1}{2}2 + \frac{1}{2}(2 + E)$. Then $E = \frac{1}{2}E_H + \frac{1}{2}E_T = \frac{1}{2}\left(3 + \frac{3}{2}E\right)$. Thus $E = 6$.

23. Consider the year beginning in March and ending in February. What months start with the same day of the week as March?

    (A) June only

    (B) September only

    (C) November only

    (D) December only

    (E) September and December

Solution: C. Treat the first day of March as 0 (mod 7). All months but February have 30 or 31 days so their passage adds 2 (mod 7) or 3 (mod 7). We obtain the days of the week of later months relative to March as: April∼3, May∼5, June∼1, July∼3, August∼6, September∼2, October∼4, November∼0, December∼2, January∼5, February∼1. So only November begins on the same day of the week.

24. Let S be a set of rational numbers such that the sum of any two elements of S is itself an element of S, and let $\left\{0, \frac{4}{3}, \frac{7}{4}, \frac{37}{12}, \frac{29}{6}, \frac{28}{3}\right\}$ be a subset of S. Which of the following might not be an element of S?

    (A) $\frac{74}{3}$

    (B) $\frac{299}{12}$

(C) $\frac{80}{3}$

(D) $\frac{85}{3}$

(E) 30

Solution: B. Observe that $\frac{4}{3} = \frac{16}{12}$ and $\frac{7}{4} = \frac{21}{12}$ generate the other positive elements of the subset, and 16 and 21 are co-prime. The largest number that cannot be generated by their sums is the their Frobenius number $21 \cdot 16 - 21 - 16 = 20 \cdot 15 - 1 = 299$. So S contains $k/12$ for all integers $k > 299$. Furthermore $\frac{74}{3} = 8(\frac{4}{3} + \frac{7}{4})$.

25. Real numbers x and y are chosen independently at random between 0 and 1 and -1 and 1, respectively. What is the probability that the complex number $(x+iy)^3$ is inside the first quadrant of the unit circle?

    (A) $\frac{\pi}{4}$

    (B) $\frac{\pi}{7}$

    (C) $\frac{\pi}{12}$

    (D) $\frac{\pi}{16}$

    (E) $\frac{\pi}{24}$

Solution: E. We know that $(x+iy)^3$ is in the unit circle if and only if $x+iy$ is in the unit circle. Dividing the unit disk into 12 equal sectors from the positive real axis, we see that $z \mapsto z^3$ sends every fourth sector to quadrant 1. Only 1 such sector has positive real part, so the area of interest is $\frac{\pi}{12}$. The total sample area is 2 so the probability is $\frac{\pi}{24}$.

26. Peter has 6 colors of paint, and wants to paint each of the 4 walls of a room a solid color such that no two adjacent walls are the same color. Peter lists all such ways to paint the 4 walls and picks one at random. What is the probability that two of the walls will be painted the same color?

    (A) 1/3

    (B) 2/5

    (C) 3/7

    (D) 1/2

    (E) 4/7

Solution: C. First, we count the number of ways to paint the 4 walls. Label the walls A, B, C, and D in counterclockwise order. There are 6 possible ways to paint wall A. If walls B and D are painted the same color, then there are 5 ways to pick this color. Wall C can then be painted any of the other 5 colors. If walls B and D are painted different colors, then there are $5 \cdot 4$ ways to pick those colors. Wall C can then be painted any of the other 4 colors. Therefore, there are $6 \cdot (5 \cdot 5 + 5 \cdot 4 \cdot 4) = 630$ ways to paint the walls of the room.

Trivially, there are $6 \cdot 5 \cdot 4 \cdot 3 = 360$ ways to paint the walls such that no two walls are the same color. Hence, the probability that two walls are the same color is $1 - \frac{360}{630} = \boxed{\frac{3}{7}}$.

27. Suppose $(x, y) = (2016, 2017), (2024, 2017), (2016, 2023)$, and $(2020, s)$ are solutions to the equation $x^2 + y^2 + ax + by = c$ where a, b, and c are real numbers. What is the maximum possible value of s?

    (A) 2015

    (B) 2016

    (C) 2020

    (D) 2024

    (E) 2025

Solution: E. The given equation describes a circle that goes through the points $A = (2016, 2017), B = (2024, 2017), C = (2016, 2023)$. Since AB is parallel to the x-axis and CA is parallel to the y-axis, $\angle CAB = 90°$. Thus BC is a diameter of the circle, i.e. the center and radius of the circle are $(2020, 2020)$ and 5 respectively. Since $(2020, s)$ is on the circle, $|s - 2020| = 5$, and the larger of the two solutions is 2025.

28. Given that $\frac{1}{7} = 0.\overline{142857}$, what is the sum of all distinct primes which divide 142857?

(A) 58

(B) 60

(C) 62

(D) 64

(E) 66

Solution: D. Since $\frac{1}{7} = 0.\overline{142857}$, we have $\frac{10^6}{7} = 142857.\overline{142857} = 142857 + \frac{1}{7}$, and thus, $142857 = \frac{10^6 - 1}{7}$. Using the difference of squares factorization along with the sum and difference of cubes factorizations, we have, $7 \cdot 142857 = 10^6 - 1 = (10^3-1)(10^3+1) = (10-1)(10^2+10+1)(10+1)(10^2-10+1) = 9 \cdot 111 \cdot 11 \cdot 91$. It is easy to factor $9 = 3^2$, $111 = 3 \cdot 37$, $11 = 11$, and $91 = 7 \cdot 13$. Thus, $7 \cdot 142857 = 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37$, i.e., $142857 = 3^3 \cdot 11 \cdot 13 \cdot 37$. Thus, the sum of the distinct prime which divide 142857 is $3 + 11 + 13 + 37 = 64$.

29. In a country there are 100 cities and some bidirectional roads, with any two cities connected by at most one road. It is known that for some pair of cities A and B, any shortest route from A to B involves 50 cities (including A and B themselves). At most how many roads can there be in the country?

(A) 199

(B) 246

(C) 1374

(D) 1424

(E) 2475

Solution: D. We will call two cities connected by a direct road "neighbors". Choose any 50-city route from A to B. There are 49 roads on this route and no more roads exist between the cities on the route - otherwise a "shortcut" not involving some of the cities would exists. Any city C not on that route can have at most three neighbors on the route - otherwise a "shortcut" using C would exists. Having in mind this restriction, it is possible for all cities not on the route to have exactly 3 neighbors on the route - just pick three consecutive cities on the route and connect them with direct roads to all cities not on the route. Finally, we can add direct roads between all pairs of non-route cities. Therefore the answer is $49 + 50 \times 3 + \binom{50}{2} = 1424$.

30. We say a number is deflected if it can be written as $k^2 - 1$ with k a positive integer. For example, $120 = 11^2 - 1$ is deflected. Find the sum of all numbers no greater than 300 with deflected factors.

    (A) 18902

    (B) 19567

    (C) 19996

    (D) 22092

    (E) 22687

Solution: C. The list of deflected numbers no greater than 300 is $3, 8, 15, 24, \ldots, 288$, but among them we only need to consider $3, 8, 35, 143$, as other deflected numbers are multiples of smaller ones. The sum of the numbers no greater than 300 with 3 or 8 as factors is $(3 + 6 + \ldots + 300) + (8 + 16 + \ldots + 296) - (24 + 48 + \ldots + 288) = 18902$. Together with those not included in the summation above, namely $35, 70, 140, 175, 245, 143, 286$, the final answer is 19996.

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