1. How many integers between 1 and 1000, inclusive, are divisible by 4 and 6 but not divisible by 26?

Solution: The answer is 77. The number of integers divisible by both 4 and 6 will be the number of integers divisible by their least common multiple: $\operatorname{lcm}(4,6) = 12$. This is precisely $\lfloor 1000/12 \rfloor = 83$. The number of integers which are divisible by 4, 6, and 26 is $\lfloor 1000/\operatorname{lcm}(4,6,26) \rfloor = \lfloor 1000/156 \rfloor = 6$. Hence, there are $83 - 6 = 77$ integers satisfying the desired properties.

2. Evaluate $\log_4(5) \cdot \log_5(6) \cdot \log_6(7) \cdot \log_7(8)$. Write your answer in simplest form.

Solution: Rewrite the logarithms using the change-of-base formula and cancel:

$$\frac{\ln(5)}{\ln(4)} \cdot \frac{\ln(6)}{\ln(5)} \cdot \frac{\ln(7)}{\ln(6)} \cdot \frac{\ln(8)}{\ln(7)} = \frac{\ln(8)}{\ln(4)} = \frac{\ln(2^3)}{\ln(2^2)} = \frac{3\ln(2)}{2\ln(2)} = \frac{3}{2}$$

3. Joan is flipping a coin which has equal probability 1/2 of being heads or tails. Joan flips the coin 5 times, with each flip being independent of the others. What is the probability that at least two of the flips are heads, given that the first flip is heads?

Solution: By the independence of flips, the probability that at least two of the flips are heads given that the first flip is a heads is simply the probability that at least one of the flips of the last four coins is a heads. This is 1 minus the probability that none of the four flips are heads. Therefore, the desired probability is

$$1 - \frac{1}{2^4} = \frac{15}{16}.$$

4. Albert and Katie are painting a room. Katie paints half the room red. Albert paints half of the unpainted area blue. Katie paints half of the unpainted area red, and so on. If this process continues infinitely, what fraction of the room will be painted red?

> **Solution:** The portions painted red occupy 1/2, 1/8, 1/32, ..., $1/2^{2k+1}$, ... of the room, so the total portion painted red occupies
>
> $$\frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{2^{2k}} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{1}{4^k} = \frac{1}{2}\cdot\frac{1}{1-1/4} = \frac{2}{3}.$$

5. Find the sum of the coefficients of the polynomial $(-5+3x)^9$.

> **Solution:** Using the binomial theorem,
>
> $$(-5+3x)^9 = \sum_{k=0}^{9}\binom{9}{k}(-5)^k 3^{9-k}x^{9-k}.$$
>
> The sum of the coefficients can be found by setting $x = 1$. But then we have
>
> $$(-5+3)^9 = (-2)^9 = -512.$$

6. Find the real number x for which the following polynomial is minimized:

$$(x-1)^2 + (x-2)^2 + (x-3)^2 + \cdots + (x-2012)^2 + (x-2013)^2.$$

> **Solution:** The minimum occurs at $x = 2014/2 = 1007$. We can rewrite the polynomial as the quadratic expression
>
> $$2013x^2 - 2(1+2+3+\cdots+2012+2013) + (1^2+2^2+3^2+\cdots+2012^2+2013^2).$$
>
> The x-coordinate for the vertex of the parabola $y = ax^2+bx+c$ is given by $-b/2a$. In this case,
>
> $$b = -2(1+2+3+\cdots+2012+2013) = -2\cdot\frac{2013\cdot 2014}{2} = -2013\cdot 2014,$$

so

$$-\frac{b}{2a} = -\frac{-2013 \cdot 2014}{2 \cdot 2013} = \frac{2014}{2} = 1007.$$

7. How many words can be made by arranging the six letters G, E, O, R, G, and E such that no two consecutive letters are the same?

Solution: We can count using inclusion exclusion. The number of arrangements with no restrictions is 6!/2!2!. Subtract the number of arrangements containing GG which is 5!/2! and the number of arrangements containing EE which is also 5!/2!. The arrangements with both GG and EE were subtracted twice, so we must add them back, which is 4!. The result is

$$\frac{6!}{2!2!} - \frac{5!}{2!} - \frac{5!}{2!} + 4! = 180 - 60 - 60 + 24 = 84.$$

8. Find the smallest real number $x > 1$ such that the decimal expansions of x^2 and $\frac{1}{x^2}$ are the same except for the position of the decimal point.

Solution: $\sqrt[4]{10}$. We know that $10^k \cdot \frac{1}{x^2} = x^2$ for some integer value of k. This means that $10^k = x^4$, so $x = 10^{k/4}$. Taking $k = 1$ gives the smallest real number satisfying the condition.

9. Evaluate

$$\frac{1}{3^2-1} + \frac{1}{5^2-1} + \frac{1}{7^2-1} + \cdots + \frac{1}{2013^2-1}.$$

Solution: For each term

$$\frac{1}{a^2-1} = \frac{1}{(a-1)(a+1)} = \frac{1}{2}\frac{(a+1)-(a-1)}{(a-1)(a+1)} = \frac{1}{2}\left(\frac{1}{a-1} - \frac{1}{a+1}\right).$$

The sum is then a telescoping series

$$\frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} - \frac{1}{6}\right) + \cdots + \frac{1}{2}\left(\frac{1}{2012} - \frac{1}{2014}\right) = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2014}\right) = \frac{503}{2014}.$$

10. Find the area of the region in the upper xy-plane bounded by the circle of radius 2 centered at the origin and the lines $x = -1$ and $x = \sqrt{3}$.

> **Solution:** Note that the graph of $y = \sqrt{4 - x^2}$ is the top half of a circle with radius 2 centered at the origin $(0, 0)$. Draw line segments from the origin to each of the points $(-1, \sqrt{3})$ and $(\sqrt{3}, 1)$. Label the three regions created A_1, A_2, A_3 as shown in the diagram. Clearly, A_1 and A_2 are each right triangles with legs 1 and $\sqrt{3}$. Thus, they each have an area of $[A_1] = [A_2] = \frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$. Also, A_3 is a sector of a circle. Since A_1 and A_2 have angles of 60° and 30° at the origin, the angle of A_3 at the origin is $180^\circ 60^\circ 30^\circ = 90^\circ$. Hence, A_3 is a quarter circle of radius 2, and has an area of $\frac{1}{4} \cdot \pi \cdot 2^2 = \pi$. Then, the total area is $[A_1] + [A_2] + [A_3] = \pi + \sqrt{3}$.

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