Georgia Tech High School Math Competition

Ciphering Test — Solutions

March 3, 2012

1. Three cars are on the road, moving in the same direction. Car A is 20 miles ahead of cars B and C, which are in the same position. If A is moving at 30 miles per hour, B at 60 mph, and C at 50 mph, how long will it take for B to be exactly halfway between A and C? (You must specify your time units in your answer.)

Solution: Let $f(t) = 20 + 30t$ be the position function for car A, $g(t) = 60t$ for car B, and $h(t) = 50t$. We want to know at what time t we have $f(t) - g(t) = g(t) - h(t)$: i.e., solve

$$20 + 30t - 60t = 60t - 50t.$$

This gives us $t = 1/2$, so B reaches the halfway point in half an hour, or 30 minutes (or 1800 seconds, or 1/48 days, or...).

2. Find the average of all the digits in the product $2^{2013} \cdot 5^{2011}$.

Solution: The product is 4×10^{2011}, written as the number 4 followed by 2011 zeros, so the average of all the digits is $\frac{4}{2012} = \frac{2}{1006} = \frac{1}{503}$.

3. Find the number of pairs of distinct subsets (A, B) such that $A \subsetneq B \subseteq \{1, 2, 3, 4, 5, 6\}$.

Solution: For each integer x in $\{1, 2, 3, 4, 5, 6\}$, there are three possibilities: $x \in A \cap B$; $x \in B \setminus A$; and $x \notin B$. Assigning each x to one of these three sets gives 3^6 possibilities for (A, B), with the caveat that we're including all the cases where $A = B$. There are 2^6 possibilities for B, so subtracting the cases where $A = B$ yields $3^6 - 2^6 = 665$.

4. Find the smallest positive integer n such that 120 divides n, n^2 is a perfect cube, and n^3 is a perfect square.

> **Solution:** We know that $2, 3, 5$ divide n, so n must be of the form $n = 2^a 3^b 5^c$ for some positive integers a, b, c. We also know that 3 divides $2a, 2b, 2c$ and that 2 divides $3a, 3b, 3c$, so 6 must divide a, b, c. Taking $a = b = c = 6$ gives the smallest possible value for $n = 2^6 \cdot 3^6 \cdot 5^6 = 30^6$.

5. Suppose the infinite sum $1 + x + x^2 + x^3 + \cdots$ converges to the value $2012x$. What is the product of all possible values of x?

> **Solution:** The geometric series converges to $1/(1 - x) = 2012x$, so $1 = 2012x - 2012x^2$, or $x^2 - x + 1/2012 = 0$. Since $x = (1 \pm \alpha)/2$, where $\alpha = \sqrt{1 - 4/2012}$, the product of the possible values of x is $\frac{1}{4}(1 - \alpha^2) = \frac{1}{4}(1 - 1 + 4/2012) = 1/2012$.

6. How many three-digit natural numbers satisfy the property that the middle digit is the sum of the first and last digits?

> **Solution:** Count the number of pairs of integers $0 < A \leq 9$, $0 \leq C \leq 9$ such that $A + C \leq 9$. This corresponds to the set of desired integers ABC, where $B = A + C$. Given A, there are $10 - A$ choices for C such that $A + C \leq 9$. Thus, we have $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45$ such integers.

7. Compute $\sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}$.

> **Solution:** Let $x = \sqrt{3 + \sqrt{3 + \sqrt{3 + \cdots}}}$. Then $x = \sqrt{3 + x}$, so $x^2 - x - 3 = 0$. By the quadratic equation, $x = \frac{1 \pm \sqrt{13}}{2}$; since x must be positive, we have $x = \frac{1 + \sqrt{13}}{2}$.

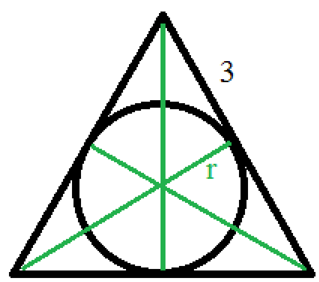
8. How many pairs of real numbers (x, y) satisfy the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{x + y}?$$

Solution: There are no solutions in $\mathbb{R}$ to this equation. Cross multiply to get $(x+y)y + (x+y)x = xy$, or $(x+y)^2 = xy$. We cannot have $xy = 0$ (or else $1/x$ or $1/y$ is undefined), and we cannot have $xy < 0$ since $(x+y)^2$ must be nonnegative. Finally, if $xy > 0$, then we have $x^2 + 2xy + y^2 = xy$, or $x^2 + y^2 = -xy$, which is also impossible since x^2 and y^2 must be positive.

9. A circle is inscribed inside an equilateral triangle so that each side of the triangle is tangent to circle. If the triangle has area $9\sqrt{3}$, find the area of the circle.

Solution: First, using the formula for the area of an equilateral triangle we see that the triangle has side length 6. Drawing altitudes, we see they intersect at the center of the triangle, which is also the center of the circle. Any altitude cuts the others into two pieces of lengths r and $2r$, where b is the radius of the smaller circle. We now have a 30-60-90 triangle whose long leg has length 3, so the short leg has length $\sqrt{3}$. Because the short leg is also r, we see the area of the smaller circle is 3π.



10. Let A be a set of integers, and define $A^{\square}$ to be the set $\{a^2 : a \in A\}$. If A contains exactly 11 elements, find the minimum possible sum of the elements in $A^{\square}$.

Solution: We get the minimum possible sum by taking

$$A = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\},$$

so that $A^{\square} = \{0, 1, 4, 9, 16, 25\}$. The sum is then 55.

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