Georgia Tech HSMC 2010

Varsity Multiple Choice

February 27^{th}, 2010

1. A cube of SPAM defined by $S = \{(x, y, z) | 0 \leq x, y, z \leq 1\}$ is cut along the planes $x = y, y = z, z = x$. How many pieces are there? (No spam gets moved until all three cuts are made.)

   (a) 5

   (b) 6

   (c) 7

   (d) 8

   (e) 9

   Solution: (B) There are six pieces. Notice that every point in the cube is defined by a set of three inequalities such as $x < y < z$. There are 3! such possibilities for this inequality so there are $3! = 6$ pieces of spam after the cuts.

2. A box contains nine balls, labeled $1, 2, 3, \ldots, 9$. Suppose four balls are drawn simultaneously. What is the probability that the sum of the labels on the balls are even?

   (a) $\frac{1}{4}$

   (b) $\frac{11}{21}$

   (c) $\frac{34}{63}$

   (d) $\frac{35}{63}$

   (e) $\frac{5}{8}$

   Solution: (B) In order for the sum of the balls to be even, there must be an even number of odd balls. This is computed by

$$\frac{\binom{4}{0}\binom{5}{4} + \binom{4}{2}\binom{5}{2} + \binom{4}{4}\binom{5}{0}}{\binom{9}{4}} = \frac{33}{63} = \frac{11}{21}$$

3. What is the perimeter of a regular hexagon whose area is $24\sqrt{3}$ square units?

   (a) $4\sqrt{3}$

   (b) 12

(c) $8\sqrt{3}$
(d) $12\sqrt{2}$
(e) 24

Solution: (E) Recall that a regular hexagon can be broken into six identical equilateral triangles with side length equal to the side length of the regular hexagon. Since the area is $24\sqrt{3}$, each triangle has area $4\sqrt{3}$. We can deduce that the side length of the hexagon is 4. Hence, the perimeter is 24.

4. Compute $\cos \pi/12$.

   (a) $\frac{\sqrt{2}}{4}$
   (b) $\frac{\sqrt{5}}{4}$
   (c) $\frac{\sqrt{6}+\sqrt{2}}{4}$
   (d) $\frac{\sqrt{2}}{2}$
   (e) $\frac{2\sqrt{2}}{3}$

Solution: (C) Use the sum of cosines identity. We have

$$\begin{aligned} \cos \pi/12 &= \cos \pi/3 - \pi/4 \\ &= \cos \pi/3 \cos \pi/4 + \sin \pi/3 \sin \pi/4 \\ &= (1/2)(\sqrt{2}/2) + (\sqrt{3}/2)(\sqrt{2}/2) \\ &= \frac{\sqrt{6}+\sqrt{2}}{4}. \end{aligned}$$

5. What is the largest possible volume of a rectangular box whose diagonal length is 18?

   (a) 64^2
   (b) $96^{3/2}$
   (c) $108^{3/2}$
   (d) $120^{2/3}$
   (e) 1600

Solution: (C) If the diagonal length of the box is 18, then if a, b, c are the side lengths of the box, it must be that $\sqrt{a^2+b^2+c^2} = 18$. Thus $a^2+b^2+c^2 = 324$. By the AM-GM inequality it follows that $\sqrt[3]{a^2b^2c^2} \leq \frac{a^2+b^2+c^2}{3} = 324/3 = 108^{3/2}$.

6. What is $i + 2i^2 + 3i^3 + 4i^4 + \cdots + 60i^{60}$?

   (a) $60i$
   (b) 20
   (c) 30

(d) $30 - 30i$

(e) $60 + 60i$

Solution: (D) Notice that each set of four consecutive terms of the form $\{4a+1, 4a+2, 4a+3, 4a+4\}$, where a is an integer, contributes $(2-2i)$ to the sum. Since we have fifteen such sets, our sum is $30 - 30i$.

7. How many negative roots does $x^4 - 5x^3 - 4x^2 - 7x + 4 = 0$ have?

   (a) 0

   (b) 1

   (c) 2

   (d) 3

   (e) 4

   Solution: (A) The answer is zero. Notice that we can factor $x^4 - 5x^3 - 4x^2 - 7x + 4$ into $(x^2-2)^2 - x(5x^2+7)$. It is straightforward to see that this expression is positive whenever x is negative. Hence there are no negative roots.

8. For how many integers x does a triangle with side lengths 12, 25 and x have all its angles acute?

   (a) 5

   (b) 6

   (c) 7

   (d) 8

   (e) 9

   Solution: (B) Notice that in order for the angle opposite the side of length 25 to be acute, $12^2 + x^2 > 25^2$. This is satisfied for $x > 21$. In order for the angle opposite side length x to be acute, $12^2 + 25^2 > x^2$. This means that $x < 28$. This gives six possibilities for integral values of x.

9. Suppose that L is a list of positive integers, not necessarily distinct, and that the number 80 is present. The average of this set is 66. When 80 is removed, the average drops to 65. What is the largest possible number in L?

   (a) 140

   (b) 460

   (c) 844

   (d) 897

   (e) 910

Solution: (D) The smallest possible numbers in L are 1, so a set with a largest possible member will have all one's, except for the number 80 and that largest number. Let x be the number of terms in set L. Then $66x = 65(x-1)+80$. This gives $x = 15$, so $|L| = 15$. Now, the sum of all the numbers is $15(66) = 990$. Subtracting 80 and thirteen 1's leaves 897, the largest possible value of an element in L.

10. Simplify the following:

$$\frac{(10^4+64)(18^4+64)(26^4+64)(34^4+64)}{(6^4+64)(14^4+64)(22^4+64)(30^4+64)}$$

   (a) 30
   (b) 65
   (c) 100
   (d) 130
   (e) 175

Solution: (B) Use the factorization $x^4+4y^4 = [(x-y)^2+y^2][(x+y)^2+y^2]$. Now, we can factor our expression so that all but the first and last terms of the numerator and denominator cancel, giving:

$$\frac{(8^2+4)(12^2+4)\cdots(36^2+4)}{(4^2+4)(8^2+4)\cdots(32^2+4)} = \frac{36^2+4}{4^2+4} = 65$$

11. What is $1001_2 \times 1343_5$ in base 10? (Here the subscripts below 1001 and 1343 indicate their respective bases.)

   (a) 2006
   (b) 2007
   (c) 2008
   (d) 2009
   (e) 2010

Solution: (B) Here $1001_2 = 9_{10}$ and $1343_5 = 223_{10}$. The product of these numbers is 2007.

12. Two sides of an isosceles triangle are 18 and 41. Compute the area of the triangle.

   (a) 320
   (b) 360
   (c) 400
   (d) 420
   (e) 440

Solution: (B) This triangle can be bisected into two 9-40-41 right triangles. Each has area 180. Thus the total area is 360.

13. When multiplied out, $15! = 130767A368000$. Compute the missing A.

    (a) 2
    (b) 4
    (c) 6
    (d) 8
    (e) 0

    Solution: (B) Notice that this number is divisible by nine, so the sum of its digits must be divisible by nine. The sum of all the other digits is 41, thus $A = 4$.

14. Suppose that $a_1 + a_2 + \cdots + a_{90}$ is an arithmetic progression with common difference 1 whose sum is 173. What is $a_2 + a_4 + \cdots a_{90}$?

    (a) 45
    (b) 64
    (c) 99
    (d) 109
    (e) 128

    Solution: (D) Let S_e be the sum of the even terms and let S_o be the sum of the odd terms. Notice that $S_e + S_o = 173$ and $2S_0 + 45 = 173$. Thus, $S_o = 69$ and $S_e = 64 + 45 = 109$.

15. A point in the plane, both of whose rectangular coordinates are integers with absolute value less than or equal to four, is chosen at random, with all such points have an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most two units?

    (a) $\frac{13}{81}$
    (b) $\frac{15}{81}$
    (c) $\frac{3\pi}{64}$
    (d) $\frac{4\pi}{81}$
    (e) $\frac{51}{64}$

    Solution: (A) Notice that there are 13 lattice points that satisfy this property out of the possible $9 \cdot 9 = 81$ lattice points.

16. Find the ratio between the area of a square inscribed in a circle and an equilateral triangle circumscribed about the same circle.

(a) $\frac{2\sqrt{3}}{18}$

(b) $\frac{2\sqrt{3}}{9}$

(c) $\frac{\sqrt{3}}{4}$

(d) $\frac{1}{3}$

(e) $\frac{1}{2}$

Solution: (B) Suppose the radius of the circle is r. The side length of the square is $r\sqrt{2}$. In addition, the altitude of the equilateral triangle is $3r$. Recall that the area of an equilateral triangle with altitude k is $\frac{k^2\sqrt{3}}{3}$. Thus, the area of the square is $2r^2$ and the area of the triangle is $3r^2\sqrt{3}$. This gives a ratio of $\frac{2\sqrt{3}}{9}$.

17. Buzz is climbing a flight of ten stairs and with each step can go up either one or two steps. How many different ways can Buzz climb the stairs?

(a) 55

(b) 64

(c) 80

(d) 89

(e) 144

Solution: (D) There is a one-to-one correspondence between fibonacci numbers of length n and the number of ways Buzz can take one or two steps. This can also be viewed as counting the number of ways to tile a 1 by 10 rectangle using 1 by 1 squares and 1 by 2 dominoes. For more information, see the book by Arthur Benjamin and Jennifer Quinn, Proofs That Really Count: The Art of Combinatorial Proofs.

18. What is the product of the cube root of 4 times the fourth root of 8?

(a) $12^{1/7}$

(b) $2 \cdot 12^{1/7}$

(c) $32^{3/7}$

(d) $2 \cdot 2^{5/12}$

(e) 4

Solution: (D) This product is equivalent to $2^{2/3} \cdot 2^{3/4} = 2^{17/12}$.

19. Which of the following is equivalent to the statement: If P is true, then Q is false?

(a) P is true or Q is false.

(b) If Q is false, then P is true.

(c) If P is false, then Q is true.

(d) If Q is true, then P is false.

(e) If Q is true, then P is true.

Solution: (D) Statement D is the contrapositive to the statement in the question.

20. Suppose the sum of all the angles except for one of a convex polygon is 1790. The value of the angle in degrees of the other angle is:

(a) 5

(b) 10

(c) 30

(d) 60

(e) 120

Solution: (B) The sum of the angles in a convex polygon must be a multiple of 180, and each angle must be less than or equal to 180 degrees. Hence, the last angle must be 10 degrees.

21. Cheapo airlines flies from 19 cities, and from each city they have decided to offer nonstop flights to seven other cities. Nonstop service operates in both directions so if there is a nonstop flight from city A to city B, there is a nonstop flight from city B to city A. How many different ways can this airline construct its nonstop schedule?

(a) 19!

(b) 19!7!

(c) $\binom{19}{7}\binom{18}{7}\cdots\binom{7}{7}$

(d) 19^7

(e) none of these

Solution: (E) Such a flight schedule is impossible. If we model this problem as a graph, (a mathematical model using dots and lines to represent airports and non-stop routes, respectively) we see that there are $19 \cdot 7 = 133$ different nonstop flights. However, nonstop flights occur in pairs, so the total number of flights must be even. Thus, this schedule is impossible.

22. Five fair dice are rolled. What is the probability that at least four of them have either a 1 or a 2 on their top face?

(a) 3/243

(b) 5/243

(c) 9/243

(d) 11/243

(e) none of these

Solution: (D) The probability of rolling exactly four 1's or 2's is $5(1/3)^4(2/3) = 10/243$. The probability of rolling all 1's and 2's is $(1/3)^5 = 1/243$. Thus the total probability is 11/243.

23. The base four representation of x is:

$$12133221231231212312212.$$

What is the first digit (the leftmost digit) of x in base 16? (Here numbers 10-15 are represented by A through F, respectively.)

(*a*) 4
(*b*) 6
(*c*) 9
(*d*) A
(*e*) C

Solution: (B) Notice that the digits of x in base 16 are simply the base 16 representations of each pair of consecutive digits of x when they are written in base 4. Therefore, the first digit is 12 in base 4, which is 6 in base 16.

24. Six boys and eight girls go on a nature hike. Each boy will pick either zero or two flowers and each girl will pick either 0 or 3 flowers. How many ways can the group collectively pick 20 flowers?

(*a*) 168
(*b*) 479
(*c*) 863
(*d*) 1050
(*e*) 1218

Solution: (E) The generating function for this situation is given by $(1+x^2)^6(1+x^3)^8$. In order for the total sum of flowers to be twenty, it must be either that 1 or 4 boys pick two flowers. Thus, the x^{20} coefficient of this expression is given by $\binom{6}{1}\binom{8}{6} + \binom{6}{4}\binom{8}{4} = (6)(28) + (15)(70) = 1218$.

25. What is the coefficient of x^{11} in the expansion of $(1 + 2x - x^2)^6$?

(*a*) -18
(*b*) -12
(*c*) 0
(*d*) 18
(*e*) 66

Solution: (B) There are six terms of the form $2x(-x^2)^5$. This gives a total of -12 as the coefficient of x^{11}.