Georgia Institute of Technology

High School Mathematics Competition 2010

Varsity Proof-Based Test
Problem #1

ID#:
Score:

Prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} < 2.$$

Solution: Let $S_n = \sum_{k=1}^{n} \frac{1}{k^2} < 2$. For $k > 1$, as $k - 1 < k$, we have that $\frac{1}{k(k-1)} > \frac{1}{k^2}$. As a result $S_n < 1+1/(1\cdot2)+1/(2\cdot3)+1/(3\cdot4)+\cdots+1/(n\cdot(n-1))$. Notice we can rewrite $\frac{1}{k(k-1)}$ as $\frac{1}{k} - \frac{1}{k-1}$ to obtain a telescoping series. So

$$S_n < 1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) = 2 - \frac{1}{n}$$

This completes the proof.

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Problem #2

ID#:
Score:

Prove that if P is any point in an equilateral tetrahedron of height h then the sum of the perpendicular distances from P to the four faces of the tetrahedron is equal to h.

Solution: Join point P to the vertices of the tetrahedron. This creates four pyramidal regions. Consider the volume of each pyramid. The formula for the volume of a pyramid is $\frac{1}{3}bh$, where b denotes the area of the base of the pyramid. Notice that the base of these four pyramids is the same as the base of the entire pyramid, which also has a volume of $\frac{1}{3}bh$. Since the volume of the sum of the smaller pyramids is the same as the entire pyramid, the sum of the heights (i.e. perpendicular distances) must be equal to h. This completes the proof.

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Varsity Proof-Based Test
Problem #3

ID#:
Score:

The points of a plane are colored three colors. Show there exist two points with distance one that both have the same color.

Solution: Let the colors be black, white and red. Suppose that any two points with distance one have different colors. Choose any red point r and assign to it the figure below. One of the two points b and w must be white and the other black. Hence, the point r' must be red. Notice that if we rotate the figure about r we get a circle of red points r'. This circle contains a chord of length 1, a contradiction.

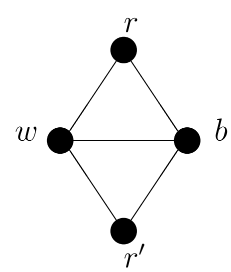

Figure 1: The figure.

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Varsity Proof-Based Test
Problem #4

ID#:
Score:

Let x, y and z be nonnegative real numbers such that $x+y+z=1$. Show that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 64.$$

Solution: This question uses extensive use of the AM-GM inequality. In particular, by the AM-GM inequality we have $\frac{1}{3} = \frac{x+y+z}{3} \geq \sqrt[3]{xyz}$. Therefore $xyz \leq (1/3)^3 = 1/27$. Consider the expansion of the left hand side of our inequality. We have

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) = 1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{xy}+\frac{1}{xz}+\frac{1}{yz}+\frac{1}{xyz}.$$

We can use the AM-GM inequality to bound the sum of the other terms as well. In particular we have,

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{3}{\sqrt[3]{xyz}} \geq 3 \cdot \sqrt[3]{27} = 9$$

$$\frac{1}{xy}+\frac{1}{yz}+\frac{1}{yz} \geq \frac{3}{\sqrt[3]{x^2y^2z^2}} \geq 3 \cdot \sqrt[3]{27^2} = 27$$

So summing these estimates we know that

$$\left(1+\frac{1}{x}\right)\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right) \geq 1+9+27+27 = 64$$

We obtain equality when $x=y=z=1/3$.

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Varsity Proof-Based Test
Problem #5

ID#:

Score:

Let α be a positive irrational number. Prove that the two sequences

$$\lfloor 1+\alpha \rfloor, \lfloor 2+2\alpha \rfloor, \lfloor 3+3\alpha \rfloor, \ldots, \lfloor n+n\alpha \rfloor, \ldots$$

and

$$\lfloor 1+\alpha^{-1} \rfloor, \lfloor 2+2\alpha^{-1} \rfloor, \lfloor 3+3\alpha^{-1} \rfloor, \ldots, \lfloor n+n\alpha^{-1} \rfloor, \ldots$$

together contain every positive integer exactly once.

Solution: Observe that $\frac{1}{1+\alpha} + \frac{1}{1+1/\alpha} = 1$. The problem will follow if we can prove the following lemma: Let α, β be positive irrational real numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Then their sequences, $a_n = \lfloor \alpha n \rfloor, b_n = \lfloor \beta n \rfloor$ are disjoint and their union is $\mathbb{N}$.

Proof of lemma: First, suppose that an integer is in both sequences. Then $\lfloor \alpha m \rfloor = \lfloor \beta n \rfloor = q$, for some q. This implies that $q < \alpha m < q+1, q, \beta n < q+1$. Since, α, β are irrational, $\frac{m}{q+1}, \frac{1}{\alpha} < \frac{m}{q}$, $\frac{n}{q+1} < \frac{1}{\beta} < \frac{n}{q}$. If we add these two inequalities, we have $\frac{m+n}{q+1} < 1 < \frac{m+n}{q}$. This implies that $m+n < q+1, q < m+n$. So $q < m+n < q+1$ This is a contradiction, as m, n are integers.

Now we will show that every integer is included. Notice that at least one of $\alpha, \beta \subset (1, 2]$, because if both were larger than two, then $1/\alpha + 1/\beta < 1$, a contradiction. Suppose that there exists some interval $(q, q+1)$, that does not contain an element of either sequence. This implies that $\alpha m < q < q+1 < \alpha(m+1)$ and $\beta n < q < q+1 < \beta(n+1)$. So $\frac{m}{q} < \frac{1}{\alpha} < \frac{m+1}{q+1}$ and $\frac{n}{q} < \frac{1}{\beta} < \frac{n+1}{q+1}$. If we add these two inequalities, we get $\frac{m+n}{q} < 1 < \frac{m+n+2}{q+1}$. So $m+n < q < q+1 < m+n+2$. This is a contradiction, as there is no way to obtain two successive integers between $m+n$ and $m+n+2$. This completes the proof.