Georgia Tech HSMC 2010

Junior Varsity Multiple Choice

February 27^{th}, 2010

1. A box contains nine balls, labeled $1, 2, 3, \ldots, 9$. Suppose four balls are drawn simultaneously. What is the probability that the sum of the labels on the balls are even?

   (a) $\frac{1}{4}$
   (b) $\frac{11}{21}$
   (c) $\frac{34}{63}$
   (d) $\frac{35}{63}$
   (e) $\frac{5}{8}$

   <u>Solution:</u> (B) In order for the sum of the balls to be even, there must be an even number of odd balls. This is computed by

   $$\frac{\binom{4}{0}\binom{5}{4} + \binom{4}{2}\binom{5}{2} + \binom{4}{4}\binom{5}{0}}{\binom{9}{4}} = \frac{33}{63} = \frac{11}{21}$$

2. What is the perimeter of a regular hexagon whose area is $24\sqrt{3}$ square units?

   (a) $4\sqrt{3}$
   (b) 12
   (c) $8\sqrt{3}$
   (d) $12\sqrt{2}$
   (e) 24

   <u>Solution:</u> (E) Recall that a regular hexagon can be broken into six identical equilateral triangles with side length equal to the side length of the regular hexagon. Since the area is $24\sqrt{3}$, each triangle has area $4\sqrt{3}$. We can deduce that the side length of the hexagon is 4. Hence, the perimeter is 24.

3. What is the slope of the line $\frac{x}{3} + \frac{y}{2} = 1$?

   (a) $\frac{-3}{2}$
   (b) $\frac{-2}{3}$

(c) $\frac{1}{3}$
(d) $\frac{2}{3}$
(e) $\frac{3}{2}$

Solution: (B) Rewriting this expression, we get $y = \frac{-2}{3}x + 2$. Thus, the slope is -2/3.

4. A man walks one mile east, then one mile northeast, then another mile east. Find the distance, in miles, between the man's initial and final positions.

   (a) $2\sqrt{2}$
   (b) $\sqrt{3+2\sqrt{2}}$
   (c) $\sqrt{5+2\sqrt{2}}$
   (d) $\sqrt{15}$
   (e) $\sqrt{11+\sqrt{2}}$

   Solution: (C) The man is $2+\frac{\sqrt{2}}{2}$ miles east of his original position and $\frac{\sqrt{2}}{2}$ miles north of his original position. By the Pythagorean theorem, the distance between the man's initial and final positions is $\sqrt{5+2\sqrt{2}}$.

5. What is the largest possible volume of a rectangular box whose diagonal length is 18?

   (a) 64^2
   (b) $96^{3/2}$
   (c) $108^{3/2}$
   (d) $120^{2/3}$
   (e) 1600

   Solution: (C) If the diagonal length of the box is 18, then if a, b, c are the side lengths of the box, it must be that $\sqrt{a^2+b^2+c^2} = 18$. Thus $a^2+b^2+c^2 = 324$. By the AM-GM inequality it follows that $\sqrt[3]{a^2b^2c^2} \leq \frac{a^2+b^2+c^2}{3} = 324/3 = 108^{3/2}$.

6. For how many integers x does a triangle with side lengths 12, 25 and x have all its angles acute?

   (a) 5
   (b) 6
   (c) 7
   (d) 8
   (e) 9

   Solution: (B) Notice that in order for the angle opposite the side of length 25 to be acute, $12^2 + x^2 > 25^2$. This is satisfied for $x > 21$. In order for the angle opposite side length x to be acute, $12^2 + 25^2 > x^2$. This means that $x < 28$. This gives six possibilities for integral values of x.

7. Suppose that L is a list of positive integers, not necessarily distinct, and that the number 80 is present. The average of this set is 66. When 80 is removed, the average drops to 65. What is the largest possible number in L?

   (a) 140
   (b) 460
   (c) 844
   (d) 897
   (e) 910

   Solution: (D) The smallest possible numbers in L are 1, so a set with a largest possible member will have all one's, except for the number 80 and that largest number. Let x be the number of terms in set L. Then $66x = 65(x-1)+80$. This gives $x = 15$, so $|L|$ $= 15$. Now, the sum of all the numbers is $15(66) = 990$. Subtracting 80 and thirteen 1's leaves 897, the largest possible value of an element in L.

8. What is $1001_2 \times 1343_5$ in base 10? (Here the subscripts below 1001 and 1343 indicate their respective bases.)

   (a) 2006
   (b) 2007
   (c) 2008
   (d) 2009
   (e) 2010

   Solution: (B) Here $1001_2 = 9_{10}$ and $1343_5 = 223_{10}$. The product of these numbers is 2007.

9. Brian just won the lottery and received an award of "50,000 yolts a year for life". Unfortunately, inflationary pressures in Brian's country means that the buying power of his 50,000 yolts decreases by ten percent each year. Brian can take either the 50,000 yolts a year for life, or a lump sum payment of x yolts today. How large would the lump sum payment have to be such that both offers would have the same net present value? (That is, so that Brian would be able to generate the same amount of money per year from an initial investment of x assuming he can earn interest at the bank at the rate of inflation.)

   (a) 500,000
   (b) 550,000
   (c) 600,000
   (d) 1,000,000
   (e) 1,200,000

Solution: (A) Brian's buying power can be described by the geometric series,

$$50,000(1 + .9 + .9^2 + .9^3 + \cdots) = 50,000(10) = 500,000$$

10. My favorite store was having a sale on shoes. The pair I wanted cost \$70.00, but was discounted by 30 percent. There was 10 percent sales tax, taken after the discount. How much did I pay for my shoes?

    (a) \$49.00
    (b) \$52.50
    (c) \$53.90
    (d) \$56.30
    (e) \$57.60

    Solution: (C) After the discount, the shoes cost \$49.00. An additional \$4.90 in tax, brings to total to \$53.90.

11. Two sides of an isosceles triangle are 18 and 41. Compute the area of the triangle.

    (a) 320
    (b) 360
    (c) 400
    (d) 420
    (e) 440

    Solution: (B) This triangle can be bisected into two 9-40-41 right triangles. Each has area 180. Thus the total area is 360.

12. When multiplied out, $15! = 130767A368000$. Compute the missing A.

    (a) 2
    (b) 4
    (c) 6
    (d) 8
    (e) 0

    Solution: (B) Notice that this number is divisible by nine, so the sum of its digits must be divisible by nine. The sum of all the other digits is 41, thus $A = 4$.

13. What is one fifth of one half of one quarter of 40?

    (a) 1/2
    (b) 1
    (c) 2

(*d*) 3

(*e*) 4

Solution: (B) $40(.2)(.25)(.5) = 1$.

14. How many ways can three board members be elected from a class of 15 students?

(*a*) 3^3

(*b*) 450

(*c*) 455

(*d*) 700

(*e*) 840

Solution: (C) There are $\binom{15}{3}$ possibilities. This is equal to 455.

15. Suppose that $a_1 + a_2 + \cdots + a_{90}$ is an arithmetic progression with common difference 1 whose sum is 173. What is $a_2 + a_4 + \cdots a_{90}$?

(*a*) 45

(*b*) 64

(*c*) 99

(*d*) 109

(*e*) 128

Solution: (D) Let S_e be the sum of the even terms and let S_o be the sum of the odd terms. Notice that $S_e + S_o = 173$ and $2S_0 + 45 = 173$. Thus, $S_o = 69$ and $S_e = 64 + 45 = 109$.

16. A point in the plane, both of whose rectangular coordinates are integers with absolute value less than or equal to four, is chosen at random, with all such points have an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most two units?

(*a*) $\frac{13}{81}$

(*b*) $\frac{15}{81}$

(*c*) $\frac{13}{64}$

(*d*) $\frac{4\pi}{81}$

(*e*) $\frac{51}{64}$

Solution: (A) Notice that there are 13 lattice points that satisfy this property out of the possible $9 \cdot 9 = 81$ lattice points.

17. How many license plates consist of one number followed by two letters followed by four numbers?

(a) 10^7

(b) $10^5 26^2$

(c) $\binom{26}{2}\binom{10}{5}$

(d) $26!10!$

(e) 2^{21}

Solution: (B) There are ten choices for each number and 26 choices for each letter. These choices are independent so we multiply them, giving choice B.

18. Find the ratio between the area of a square inscribed in a circle and an equilateral triangle circumscribed about the same circle.

(a) $\frac{2\sqrt{3}}{18}$

(b) $\frac{2\sqrt{3}}{9}$

(c) $\frac{\sqrt{3}}{4}$

(d) $\frac{1}{3}$

(e) $\frac{1}{2}$

Solution: (B) Suppose the radius of the circle is r. The side length of the square is $r\sqrt{2}$. In addition, the altitude of the equilateral triangle is $3r$. Recall that the area of an equilateral triangle with altitude k is $\frac{k^2\sqrt{3}}{3}$. Thus, the area of the square is $2r^2$ and the area of the triangle is $3r^2\sqrt{3}$. This gives a ratio of $\frac{2\sqrt{3}}{9}$.

19. How many prime numbers are there under 30?

(a) 10

(b) 11

(c) 12

(d) 13

(e) 14

Solution: (A) The prime numbers under 30 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

20. A positive number x satisfies the inequality $\sqrt{x} < 2x$ if and only if

(a) $x > 1/4$

(b) $x > 2$

(c) $x > 4$

(d) $x < 1/4$

(e) $x < 4$

Solution: (A) Since x is positive $\sqrt{x} < 2x$ is equivalent to $x < 4x^2$. This simplifies to $1 < 4x$. This means the correct answer is choice A.

21. Buzz is climbing a flight of ten stairs and with each step can go up either one or two steps. How many different ways can Buzz climb the stairs?

    (a) 55

    (b) 64

    (c) 80

    (d) 89

    (e) 144

    Solution: (D) There is a one-to-one correspondence between fibonacci numbers of length n and the number of ways Buzz can take one or two steps. This can also be viewed as counting the number of ways to tile a 1 by 10 rectangle using 1 by 1 squares and 1 by 2 dominoes. For more information, see the book by Arthur Benjamin and Jennifer Quinn, Proofs That Really Count: The Art of Combinatorial Proofs.

22. Evaluate the following sum:

    $$12 + 15 + 18 + \cdots + 1002$$

    (a) 139659

    (b) 156744

    (c) 167817

    (d) 189463

    (e) 231006

    Solution: (C) This is an arithmetic series with first term 12 and common difference 3. So $1002 = 12 + 3(n-1)$, and $n = 331$. Thus the sum of the series is $(331/2)(12+1002) = 167817$.

23. What is the product of the cube root of 4 times the fourth root of 8?

    (a) $12^{1/7}$

    (b) $2 \cdot 12^{1/7}$

    (c) $32^{3/7}$

    (d) $2 \cdot 2^{5/12}$

    (e) 4

    Solution: (D) This product is equivalent to $2^{2/3} \cdot 2^{3/4} = 2^{17/12}$.

24. Cheapo airlines flies from 19 cities, and from each city they have decided to offer nonstop flights to seven other cities. Nonstop service operates in both directions so if there is a nonstop flight from city A to city B, there is a nonstop flight from city B to city A. How many different ways can this airline decide on which nonstop cities fly from each airport?

(a) 19!

(b) 19!7!

(c) $\binom{19}{7}\binom{18}{7}\cdots\binom{7}{7}$

(d) 19^7

(e) none of these

Solution: (E) Such a flight schedule is impossible. If we model this problem as a graph, (a mathematical model using dots and lines to represent airports and non-stop routes, respectively) we see that there are $19 \cdot 7 = 133$ different nonstop flights. However, nonstop flights occur in pairs, so the total number of flights must be even. Thus, this schedule is impossible.

25. The base four representation of x is:

$$1213322123123121231222 12.$$

What is the first digit (the leftmost digit) of x in base 16? (Here numbers 10-15 are represented by A through F, respectively.)

(a) 4

(b) 6

(c) 9

(d) A

(e) C

Solution: (B) Notice that the digits of x in base 16 are simply the base 16 representations of each pair of consecutive digits of x when they are written in base 4. Therefore, the first digit is 12 in base 4, which is 6 in base 16.