Problem #1

The perimeter of parallelogram $ABCD$ is 30 and its altitudes are length 4 and 7. What is $\sin A$?

Solution: There are two ways we can obatin the area of $ABCD$. Let x be one side's length. Let $15-x$ be the other side length. The area is equal to both $7x$ and $4(15-x)$. Equating these two expressions gives $x = 60/11$. Then $\sin A = 4/x = 4/(60/11) = 11/15$. Notice that the sine of each angle is the same since the sine function is symmetric about 90 degrees.

Answer:

Problem #2

The number $(9^6 + 1)$ is the product of three primes. What is the largest of these three primes?

Solution: Notice that $(9^6 + 1) = (9^2 + 1)(9^4 - 9^2 + 1) = 82 \cdot 6481 = 2 \cdot 41 \cdot 6481$. So the answer is 6481.

Answer:

PROBLEM #3

Seven 3-element subsets of a finite set are selected so that for any two elements of the set, there exists exactly one among the seven subsets that contains both of them. Find the number of elements in the set.

Solution: Suppose there are n elements of the set. Thus there are $\binom{n}{2} = (n(n-1)/2)$ 2-element subsets of the set. A given 3-element subset contains three 2-element sets. Thus there are $n(n-1)/2 = 3 \cdot 7 = 21$. Solving for n gives $n = 7$.

Answer:

Problem #4

What is the units digit of $9^{12}13^{13}17^{14}$?

Solution: Notice that 9^{12} ends in a 1 and $13 \cdot 17$ ends in a 1. Thus some integer ending in 1 is multiplied by 17. This gives a units digit of 7.

Answer:

Problem #5

A pair of fair dice is rolled. Given that neither die shows a "6", what is the probability that the sum of the dice is 7?

Solution: There are 25 possibilities for rolls that do not involve a six, and four of these possibilities give a seven. Thus the probability is 4/25 or .16.

Answer:

Problem #6

How many positive integers less than 180 are relatively prime to 180?

Solution: This is asking for Euler's phi function of 180. This is computable via $180(1 - 1/2)(1 - 1/3)(1 - 1/5) = 48$, as 2, 3 and 5 are the only prime factors of 180.

Answer:

Problem #7

Let S be the set of all 17-digit positive integers where leading zeroes are allowed. An integer is chosen at random from S. The probability that it is a palindrome (reads the same forwards as backwards) is $1/10^k$. What is k?

Solution: The answer is 8. Notice that the middle number can be anything, however for every set of first eight digits, there is only one corresponding set of last eight digits which makes the number a palindrome.

Answer:

Problem #8

Evaluate the sum $6 - 4 + \frac{8}{3} - \frac{16}{9} + \frac{32}{27} - \cdots$.

Solution: This is a geometric series with first term 6 and common ratio -2/3. Thus, the value of this geometric series is $\frac{6}{1-(-2/3)} = \frac{6}{5/3} = \frac{18}{5}$. The answer is 18/5.

Answer:

Problem #9

Triangle ABC has $\tan A = 3/4$ and $\tan(B) = 21/20$. What is AC/BC?

Solution: By the Law of Sines, we know that $\sin(A) = 3/5$ and $\sin(B) = 21/29$. So $AC/BC = (21/29)(5/3) = 35/29$. So the answer is $35/29$.

Answer:

Problem #10

The square below can be filled in so that each row and each column contains each of the numbers 1, 2, 3 and 4 exactly once. What does x equal?

			1
	2		
		x	
1			4

Solution: Notice that the bottom row is forced, then the second column is forced, then the top row, and finally the third column. This gives $x = 4$.

2	4	3	1
	2	1	
	1	x	
1	3	2	4

Answer: