Problem #1

What is the maximum area of a quadrilateral with sides 1, 4, 7, and 8?

Solution: The answer is 18. We can assume the sides 1 and 8 are neighbors. If not, we cut the quadrilateral along a diagonal and turn over one of the triangles in order to obtain a cyclic quadrilateral (a quadrilateral which a circle can be circumscribed about it so that the circle touches each vertex (see Engel, Problem Solving Strategies, chapter 8.3, p. 323-324). From this analysis, we know that the quadrilateral has area $\leq \frac{1 \cdot 8}{2} + \frac{4 \cdot 7}{2} = 18$. This can be realized with two right triangles with common hypotenuse $\sqrt{65}$.

Answer:

Problem #2

Consider a league where there are N players that form 7 teams, each pair of teams have one common member and each player is on two teams. What is the value of N, the number of players?

Solution: The answer is 21. We can view this scenario as a graph (a combinatorial structure consisting of dots (vertices) and lines (edges)). In particular consider K_7, the graph consisting of 7 dots all connected to each other. The teams are the vertices and the players are the edges. There are 21 edges in K_7, so there are 21 players.

Answer:

Problem #3

Recall that $i = \sqrt{-1}$. What is $(i+1)^5(2-2i)^4$? (Your answer should be of the form $a + bi$.)

Solution: The trick to solving this problem quickly is by factoring correctly. We get

$$\begin{aligned}(i+1)^5(2^4)(1-i)^4 &= 16(i+1)[(1+i)(1-i)]^4 \\ &= 16(i+1)(2^4) = 256 + 256i.\end{aligned}$$

The answer is $256 + 256i$.

Answer:

Problem #4

What is $\tan(10^\circ)\tan(20^\circ)\tan(30^\circ)\cdots\tan(80^\circ)$?

Solution: The answer is 1. Expanding $\tan(x)$ to $\sin(x)/\cos(x)$ and then observing that $\sin(x) = \cos(90 - x)$ for $0 \leq x \leq 90$, gives that $\tan(x) \cdot \tan(90 - x) = 1$.

Answer:

Problem #5

How many ways can 3 Georgia Tech students, 3 Georgia students, 2 Georgia State students and 2 Harvey Mudd students sit around a circular table such that students from the same school sit together?

Solution: The four schools can be arranged around the table in $(4-1)! = 6$ ways. Each group of 3 can be arranged in 3! ways and each group of 2 can be arranged in 2! ways. This gives $6^3 2^2 = 864$. Notice that it is unclear whether the positions shifted around the table or not are also different ways, so we accepted a solution of 8640 as well.

Answer:

Problem #6

If $x - \frac{1}{x} = 5$, what is $x^4 + \frac{1}{x^4}$?

Solution: The answer is 727. Square both sides of the equation $x - \frac{1}{x} = 5$ to get $x^2 + \frac{1}{x^2} = 27$. Square both sides again to get $x^4 + \frac{1}{x^4} + 2 = 27^2$. Thus, $x^4 + \frac{1}{x^4} = 729 - 2 = 727$.

Answer:

Problem #7

Twelve identical letters are to be placed into four mailboxes. How many ways can this be done?

Solution: Consider 15 containers in a line. We will make three of these containers "dividers" that partition the other twelve boxes into four groups, one for each mailbox. This is $\binom{15}{3} = 455$.

Answer:

Problem #8

Find the number of integers n such that $1 < n \leq 2009$ for which the fraction $\frac{n^2-5}{n+6}$ reduces to an integer.

Solution: Write

$$\frac{n^2-5}{n+6} = \frac{n^2-36+31}{n+6} = n-6+\frac{31}{n+6}.$$

Notice that 31 is prime and so the only value for n that works is $n = 25$. Therefore the answer is 1.

Answer:

Problem #9

What is the size of the largest subset S of $\{1, 2, \ldots, 50\}$ such that no pair of distinct elements of S has a sum divisible by 7?

Solution: Take all the numbers congruent to 1, 2,3 modulo 7, which is 22, and add one number congruent to 0 mod 7, for 23.

Answer:

Problem #10

Given a cube, determine the ratio of the volume of the octahedron formed by connecting the centers of each face of the cube (as the vertices of the octahedron) to the volume of the cube.

Solution: Notice that the base of the two pyramids that make up the octahedron are $1/2$ the area of the cross-section of the cube. So using the $\frac{1}{3}bh$ volume formula for pyramids, we have the ratio as $1/3 \cdot 1/2 = 1/6$.

Answer: