Georgia Institute of Technology

High School Mathematics Competition 2009

Junior Varsity Proof-Based Test
Problem #1

ID#:

In triangle ABC, point E is on $\bar{AB}$, so that $AE = \frac{1}{2}EB$. Find CE if $AC = 4$, $CB = 5$, $AB = 6$.

First Solution: First, we will show a quick solution using Stewart's theorem, we have

$$(AC)^2(EB) + (CB)^2(AE) = (AB)[(CE)^2 + (AE)(EB)]$$

This gives $114 = 6(CE)^2 + 48$, so that $CE = \sqrt{11}$.

Second Solution: A more standard solution uses Heron's formula. Triangles ACE and ACB share the same altitude and $AE = \frac{1}{3}AB$, the area of triangle ACE is equal to one third the area of triangle ACB. By Heron's formula,

$$\frac{1}{3} \text{ the area of } ACB = \frac{1}{3}\sqrt{\frac{15}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2}} = \frac{5}{4}\sqrt{7}.$$

Suppose that $CE = x$. Then the area of triangle ACE is equal to

$$\sqrt{\frac{6+x}{2} \cdot \frac{6-x}{2} \cdot \frac{x+2}{2} \cdot \frac{x-2}{2}}$$

$$= \frac{1}{4}\sqrt{-(x^2-36)(x^2-4)}.$$

Setting $y = x^2$ gives us

$$\frac{5}{4}\sqrt{7} = \frac{1}{4}\sqrt{-(y^2-40y+144)}$$

.

Solving this yields $y = 11$ or 29, and after rejecting 29, we see that $CE = \sqrt{11}$.

Third Solution: By the law of cosines, $(AC)^2+(AE)^2-2(AC)(AE)\cos\alpha = (CE)^2$. Also, $(AC)^2+(AB)^2-2(AC)(AB)\cos\alpha = (CB)^2$. Here $\cos\alpha = 27/48$, and solving for $(CE)^2$ we have that $(CE)^2 = 11$. Thus $\bar{CE} = \sqrt{11}$.

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Problem #2

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After omitting three consecutive numbers from the sequence of positive integers from 1 to 1993, it was found the arithmetic mean of the remaining terms was still an integer. What three numbers were left out?

Solution: Recall that the sum of the first n positive integers is $S_n = \frac{n(n+1)}{2}$. Denote the three omitted numbers by $x-1, x$ and $x+1$. This gives the following equation, where N is an integer:

$$N = \frac{\frac{1993 \cdot 1994}{2} - (x - 1 + x + x + 1)}{1993 - 3}$$

This simplifies to $N = (1987021 - 3x)/1990$. Plugging in $x = 2$ and $x = 1992$ as upper and lower bounds for N, it is apparent that $N = 996, 997$ or 998. The only one where x is an integer occurs when $N = 997$. The omitted integers are thus 996, 997, 998.

Second Solution: We begin again with

$$N = \frac{1993 \cdot 997 - 3x}{1990}.$$

Then $x = 997 + y$, where $-995 \leq y \leq 995$. So

$$N = \frac{1990 \cdot 997 - 3y}{1990} = 997 - \frac{3y}{1990}$$

. Now $\frac{3y}{1990} = 987 - N$, an integer. But $\frac{-3}{2} \leq \frac{3y}{1990} \leq \frac{3}{2}$ implies that $\frac{3y}{1990} = \pm 1, 0$. This gives that $y = 0$, so $x = 997$ and the omitted integers are thus 996,997,998.

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Problem #3

ID#:

Determine, with proof, all values of the positive integer n for which 4^n+n^4 is prime.

Solution: First notice that if $n = 1$, the number is prime. Next observe that n must be odd. Now, suppose that $n + 1 = 2r$. Then

$$4^n + n^4 = (2^n + n^2)^2 - n^2 2^{n+1} = (2^n + n2^r + n^2)(2^n - n2^r + n^2)$$

We can check by hand when $r = 2$ or 3 for testing whether the number is composite. When $r > 3$, then $2^n + n^2 > 2^n = 2^{r-1}2^r > (2r - 1)2^r$, so both factors exceed 1.

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Problem #4

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Each vertex of a regular 7-gon is colored either red, blue, green or yellow, according to the following rules. If a vertex is colored red or blue, then the color of its first and fourth successor in clockwise orientation is neither blue nor green, and if a vertex is colored either yellow or green, the color of the first and fourth successor in clockwise orientation is neither red nor yellow. Prove that every vertex has the same color.

Solution: Usual case-analytic arguments suffice. For instance, consider vertices that are a clockwise distance three between each other. These must either both be red or blue or both be green or yellow. However, if this pair was either blue or yellow, then this would also give a color contradiction. As a result, without loss of generality this pair is colored red. Then this forces the rest of the vertices to be colored red.

Second Solution: Let $f(n)$ be the function that gives the color of vertex n. Suppose that $A = r, b, B = \{r, y\}, C = \{y, g\}$ and $D = \{b, g\}$. By our rules, we know that if $f(n) \in A$, then $f(n+1), f(n+4) \in B$ and if $f(n) \in C$, then $f(n+1), f(n+4) \in D$. As a result, if $f(n) \in B$, then $f(n+3) \in B$ and $f(n) \in D$, then $f(n+3) \in D$. So either $f(n) \in B$ for all n or $f(n) \in D$ for all n. Suppose not. Then, without loss of generality, there exists $n, n+3$ with $f(n) \in B$ and $f(n+3) \in D$. This gives a contradiction because $B \cap D = \emptyset$.

We now will show that not only that every vertex has the same color, but this color is either red or green. First, suppose that $f(n) \in B$ for all n. Then $f(n)$ is red or yellow. However, if $f(n)$ is yellow for some n, then $f(n) \in C$, so $f(n+1) \in D$, a contradiction. An analogous argument holds for when $f(n) \in D$ for all n and $f(n)$ is blue. This gives the desired result.

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Problem #5

ID#:

Twenty pairwise distinct positive integers are all less than 70. Prove that among their pairwise differences there are four equal numbers.

Solution: Denote the 20 integers a_1 to a_{20}. Then $0 < a_1 < \cdots < a_{20} < 70$. We want to prove that there is a k so that $a_j - a_i = k$ has at least four solutions. Now

$$0 < (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{20} - a_{19}) = a_{20} - a_1 \leq 68.$$

We will prove that, among the differences $a_{i+1} - a_i$, $i = 1, \ldots, 19$, there will be four equal ones. Suppose there are at most three differences equal. Then $3 \cdot 1 + 3 \cdot 2 + 3 \cdot 3 + 3 \cdot 4 + 3 \cdot 5 + 3 \cdot 6 + 7 = 70$. However, this gives a contradiction as $70 \nleq 68$.