Problem #1

Find the sum:

$$\frac{1}{3+2\sqrt{2}}+\frac{1}{2\sqrt{2}+\sqrt{7}}+\frac{1}{\sqrt{7}+\sqrt{6}}+\frac{1}{\sqrt{6}+\sqrt{5}}+\frac{1}{\sqrt{5}+2}.$$

Solution: By rationalizing the denominators, we obtain a telescoping sum with largest term 3 and smallest term 2. The difference between these terms is 1, our solution.

Answer:

Problem #2

How many integers from 1 to 1000 are divisible by 3 or 5?

Solution: This is an application of the inclusion/exclusion principle. There are 333 integers divisible by 3, 200 divisible by 5 and 66 divisible by both. This gives a total of 333 + 200 - 66 = 467.

Answer:

Problem #3

The number $aabb$ is a square greater than zero, where a and b are decimal digits. What is it?

Solution: The answer is 7744, 88^2.

Answer:

Problem #4

The area of a circle circumscribed about a regular hexagon is 200π. What is the area of the hexagon?

Solution: The radius of the circle is $10\sqrt{2}$, so the hexagon is composed of 6 equilateral triangles each of side length $10\sqrt{2}$. The area is $\frac{6(10\sqrt{2})^2\sqrt{3}}{4} = 300\sqrt{3}$.

Answer:

Problem #5

Find the exact value of $1000 + 1001 + 1002 + \cdots + 1999$.

Solution: The answer is 1499500. This is $(1000 \cdot 1000) + (999 * 500) = 1499500$.

Answer:

Problem #6

Find all real numbers k such that the equation $x^2+kx+k=0$ has only one (distinct) real solution.

Solution: Using the quadratic formula, notice that $\sqrt{k^2-4k}=0$ when $k=0$ or $k=4$. Thus the solution is $k=0$ and 4.

Answer:

Problem #7

Compute the exact value of the infinite series

$$1 - \frac{4}{3} + \frac{1}{3^2} - \frac{4}{3^3} + \frac{1}{3^4} - \frac{4}{3^5} + \cdots$$

.

Solution: We can split this expression into two infinite series and sum them. The first has sum $\frac{3}{2}$ and the second has sum $5\left(\frac{-3}{8}\right) = \frac{-15}{8}$ for a total of $\frac{-3}{8}$.

Answer:

Problem #8

The expression $(x^3 + 3x^2 + 3x + 1)^4$ is multiplied out completely to obtain a polynomial $p(x)$. Find the sum of the coefficients of $p(x)$.

Solution: Notice that $x^3 + 3x^2 + 3x + 1 = (x + 1)^3$. Then $(x^3 + 3x^2 + 3x + 1)^4 = (x + 1)^{12}$. The coefficients of this expression are the 12th row of Pascal's triangle, whose total sum is $2^{12} = 4096$.

Answer:

Problem #9

What is the remainder when $x^{100} + x^{98} + \cdots + x^4 + x^2 + 1$ is divided by $x^2 - 1$?

Solution:

$$x^{100} + x^{98} + \cdots + x^4 + x^2 + 1 =$$
$$(x^{100} - 1) + (x^{98} - 1) + \cdots + (x^2 - 1) + 50 + 1.$$

Notice that $x^{2n} - 1$ is divisible by $x^2 - 1$. As a result, the remainder is 51.

Answer:

Problem #10

An isosceles triangle has a base of length 60 and its remaining two sides have length 50. Find the side length of the square inscribed in the triangle such that two of the vertices of the square lie on the base of the triangle.

Solution: Let x be the side length of a square. We know the altitude of the isosceles triangle is length 40. After drawing the square inscribed in the triangle and the altitude, we have four similar triangles drawn inside the triangle. This gives the equation:

$$\frac{40 - x}{\frac{x}{2}} = \frac{x}{30 - \frac{x}{2}}$$

Solving for x gives $x = 24$.

Answer: