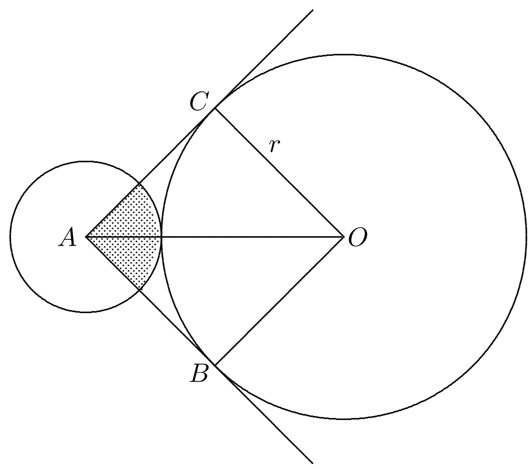
Georgia Tech HSMC 2008

Varsity Multiple Choice

February 23^{rd}, 2008

1. In the figure below, assume that the circles are mutually tangent and that the circle with center at A has radius 1. The lines AB and AC are tangent to the circle with center at O in the points B and C respectively. If the area of the shaded region is $\pi/4$, what is the radius of the circle centered at O?



$(d)\ r = \sqrt{2}+1$

2. What is the sum of the reciprocals of all the triangular numbers. (Note: An integer n is said to be a triangular number if there is a number k such that $n = 1+2+3+\cdots+k$).

$(d)\ 2$

3. Five students tried to find the sum of the first 21 positive primes and all got different solutions. Tony got 707, Pat got 709, Lee got 711, Sandy got 712, and Dale got 713. One of them is correct, Which one?

(d)Sandy

4. Let Γ be a circle with center O. Let T be a point on the circle, and P a point outside the circle such that PT is tangent to Γ. Assume that the segment OP intersects Γ in a point Q. If $PT = 12$ and $PQ = 8$, find the radius of Γ.

$(b)\ r = 5.$

5. If one picks $n \leq 26$ letters out of an alphabet of 26 letters, what is the probability that they are in alphabetical order?

$(b)\ (n!)^{-1}$

6. What is the remainder when $2^{3000} \cdot 3^{2000}$ is divided by 7?

$(b)\ 2$

7. The parabola $y = ax^2+bx+1$ has its maximum at $(2,2)$. Which of the following is correct?

$(e)\ b = 1.$

8. How many pairs of positive integers (n,m), with $n \leq m$ satisfy the equation $mn = 4(n+m)$?

$(d)\ 3$

9. Find x if $(1+\tan 1°)(1+\tan 2°)\cdots(1+\tan 45°) = 2^x$.

$(c)\ x = 23$

10. The expression $\dfrac{x+y+|x-y|}{2}$ is equivalent to...

$(a)\ \max(x,y).$

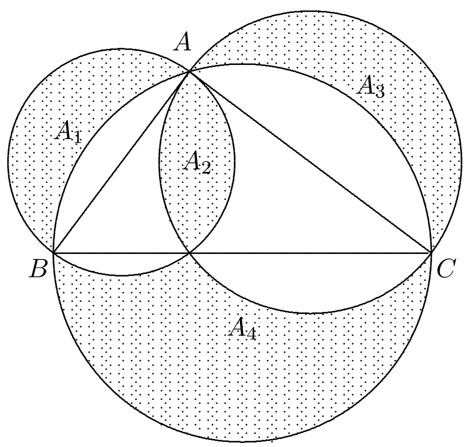
11. Let x and y be two random points in the interval $[0,1]$. What is the probability that $|x-y| < \frac{1}{2}$?

$(e)\ \frac{3}{4}$

12. How many 3-digit numbers are there in which the first digit is greater than both the second and the last digits?

$(c)\ 2\binom{10}{3} + \binom{10}{2} = 285$

13. In the image below, $\triangle ABC$ is right at A. A_1, A_2, A_3 and A_4 denote the areas of the respective shaded regions. Which of the following relations is correct?

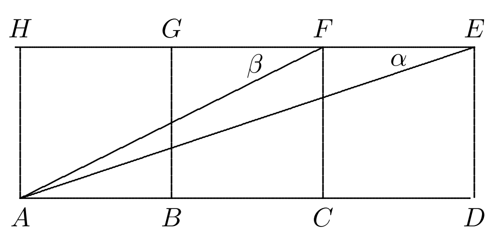


$(a)\ A_4 = A_1 + A_2 + A_3$

14. In a league with 6 teams, each team plays the others exactly once. Each game ends in either a win or a loss (no ties are allowed). What's the greatest number of teams that can end with a losing record (i.e., having lost more games than they won)?

(e) 5 teams

15. In the following picture, $ABGH$, $BCFG$ and $CDEF$ are all squares with the same side length. If $\alpha = \sphericalangle AEH$ and $\beta = \sphericalangle AFH$, which of the following statments is true?



$(b)\alpha + \beta = \dfrac{\pi}{4}$

16. Each year a school organizes an essay contest. The five laureates, namely Anna, Bart, Carla, David, and Edna, have just received their result. Each of them knows his/her own place in the list (numbered 1 to 5, without ties). Carla knows that David ended two places before Anna. Because she supposes Edna could not possibly win the contest (and she is right), she is able to derive the whole list. In which place did Carla end?

$(c)3^{\text{rd}}$

17. If $a+b+c=7$ and $\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{a+c}=\dfrac{7}{10}$ find the value of $\dfrac{c}{a+b}+\dfrac{a}{b+c}+\dfrac{b}{a+c}$.

$(c)\frac{19}{10}$

18. Inside a square you draw a smaller square and a triangle. What is the largest number of regions into which you divide the larger square?

(c) 14

19. What is the largest hypotenuse a triangle can have if all its sides have integer lengths, two of its sides are consecutive integers, and the sum of the lengths of all the sides does not exceed 2008?

(b) 925

20. Calculate

$$\frac{1}{1+\sqrt{2}+\sqrt{3}}+\frac{1}{1-\sqrt{2}+\sqrt{3}}+\frac{1}{1+\sqrt{2}-\sqrt{3}}+\frac{1}{1-\sqrt{2}-\sqrt{3}}.$$

$(e)2$