Georgia Institute of Technology

High School Mathematics Competition 2008

Junior Varsity Proof-Based Test
Problem #1

The 4^{th} century general Sun Tsu Suan-Ching wanted to know the precise number of soldiers in his regiment, but all records were destroyed after a surprise attack. He knew he had four battalions, each consisting of anywhere between 500 and 1000 soldiers. He called his regiment and ordered the soldiers to form groups of 7 men each, but 6 men were left over. Then he ask them to form groups of 8, but there were 7 men left over. With groups of 9 men, there were 8 men left over. With groups of 10 men, there were 9 left over. Then he realized he had enough information to compute the exact number of soldiers. How many soldiers were in the regiment?

Solution: Assume that n is the number of soldiers in General's Sun Tsu Suan-Ching regiment, then, if there was one more man (that's $n + 1$ men total) one could form groups of either 7, 8, 9 or 10 men without any man left over. Thus $n+1$ is divisible by each of those numbers, thus, by its minimum common multiple, that is 2520. Now, as the regiment has four batallions of at most 1000 men each, $n + 1 \leq 4001 < 5040 = 2 \cdot 2520$, thus, $n + 1 = 2520$ and therefore there are 2519 men in the regiment.

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Problem #2

What is the smallest integer number n such that a cube with edge length n can be subdivided in 2008 cubes, each of integer edge length, and how can that subdivision be made?

Solution: Note that $12^3 < 2008 < 13^3$, and as each cube has at least volume 1, such n is at least 13. There are three different ways on which this can be done. Note that

$$\begin{aligned} 13^3 = 2197 &= 3 \cdot 4^3 + 2005 \cdot 1^3 \\ &= 7 \cdot 3^3 + 1 \cdot 2^3 + 2000 \cdot 1^3 \\ &= 27 \cdot 2^3 + 1981 \cdot 1^3 \end{aligned}$$

on each case, there are 2008 cubes, and it is possible to assamble them into a cube.

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Note: Although it is not required for the problem (which indeed requires to find only one way to subdivide the cube), for self containment, we provide a proof these are indeed the only three solutions.

Note first that no cube can have edge 6 or larger, because if so, you would have a cube of volume at least 216 and 2007 of volume at least 1, so your volume is at least 2223, what already surpassed 2197. On the other hand, note that you have the following system of Diophantine equations (and need solutions with nonnegative entries)

$$\begin{cases} 5^3a + 4^3b + 3^3c + 2^3d + e = 2197 \\ a + b + c + d + e = 2008. \end{cases}$$

Convining these two solutions, we get $124a + 63b + 26c + 7d = 189$. First, note that if $a = 1$, then if $b = 1$ then $26c + 7d = 2$, what is impossible. Now, if $b = 0$ then $26c + 7d = 65$ and this equation has no solution in the positive integers. Thus $a = 0$.

If $b > 0$, note that $b = 3$, $c = d = 0$ yeld a solution. If $b = 2$ or $b = 1$, then $26c + 7d = 63$ or $26c + 7d = 126$ respectively, but neither of these equations have solutions in the positive integers.

If $a = b = 0$ then $26c + 7d = 189$, and this equation has two solutions, $c = 0,\ d = 27$ and $c = 7,\ d = 1$.

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Problem #3

Consider the unit square $ABCD$. Let E, F, G and H be points in the interior of the sides AB, BC, CD and DA respectively. Show that the perimeter of $EFGH$ is at least $2\sqrt{2}$.

<u>**Solution:**</u> (**Algebraically**) Note that by the Pythagorean Theorem, one has $EF^2 = EB^2 + BF^2$, and one also have that

$$\left(\frac{EB + BF}{2}\right)^2 \leq \frac{EB^2 + BF^2}{2},$$

thus, $\sqrt{2}EF \geq EB+BF$. Similarly, $\sqrt{2}FG \geq FC+CG$, $\sqrt{2}GH \geq GD+DH$ and $\sqrt{2}HE \geq HA + AE$. Therefore

$$\begin{aligned} EF + FG + GH + HE &\geq \frac{AE + EB + BF + FC + CG + GD + DH + HA}{\sqrt{2}} \\ &= 2\sqrt{2} \end{aligned}$$

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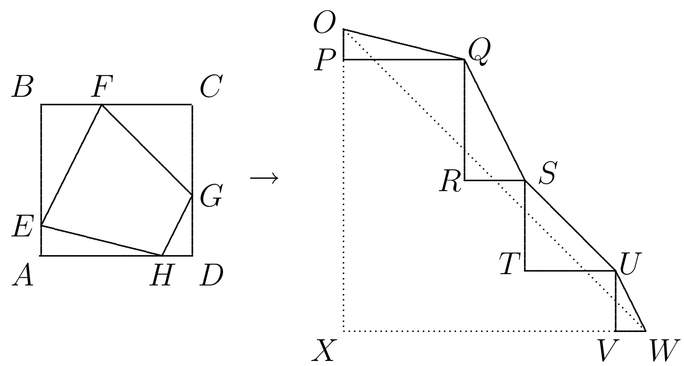
(**Geometrically**) Construct $\triangle OPQ \cong \triangle EAH$. Construct $\triangle QRS \cong \triangle EBF$ such that $QR \perp QP$ and the points P and S are on different sides of QR. Similarly construct $\triangle STU \cong \triangle GCF$ such that $RS \perp ST$ and the points R and U are on different sides of ST and $\triangle UVW \cong \triangle GDH$ such that $TU \perp UV$ and the points T and W are on different sides of UV.



Let X be the intersections of the lines OP and VW. Note that as $OP \perp PQ$ and $PQ \perp QR$, then $OP \parallel QR$. Similarly $OP \parallel QR \parallel ST \parallel UV$ and $PQ \parallel RS \parallel TU \parallel VW$. From here we can conclude that $OX \perp XW$ and also that $OX = OP+QR+ST+UV = AE+EB+CG+GD = AB+CD = 2$, and identically $XW = 2$. By the Pythagorean theorem, $OW = 2\sqrt{2}$, and by triangular inequality one has that

$$2\sqrt{2} = OW \leq OQ + QS + SU + UW = HE + EF + FG + GH.$$

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Problem #4

Show that if a, b and c are positive real numbers, then

$$1 < \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} < 2.$$

Solution: First, note that as a, b and c are all positive real numbers, then,

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} > \frac{a}{a+b+c} + \frac{b}{b+c+a} + \frac{c}{c+a+b} = \frac{a+b+c}{a+b+c} = 1.$$

Thus proving the left hand side inequality. Note that in the exact same way one can prove that

$$\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a} > 1,$$

but notice also that

$$\left(\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}\right) + \left(\frac{b}{a+b} + \frac{c}{b+c} + \frac{a}{c+a}\right) = 3,$$

and the sum inside each pair of parentheses is greater than one, thus, each of them is less than 2, proving that

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} < 2.$$

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Problem #5

If you have the numbers 1 through 10 written in 10 pieces of paper (one number on each piece). How many ways are there to write the numbers 1 to 10, one on each piece of paper (on the unwritten side) such that the difference (in absolute value) of the numbers on each side is less than 2?

Solution: Let's call $b_1, b_2, \ldots, b_{10}$ the second number written on the piece of paper where $1, 2, \ldots, 10$ were written originally (in the same respective order). Note that b_{10} is either 9 or 8 (otherwise the difference would be at least 2). If $b_{10} = 10$ then, one has to solve the same problem but with the numbers from 1 to 9 instead. Now, if $b_{10} = 9$, then $b_9 = 10$ and thus, you would have to solve the same problem with the numbers 1 to 8. Call A_k the general solution of the problem from 1 to k, then, we have that $A_{k+1} = A_k + A_{k-1}$. Also, $A_1 = 1$ (as there would be only one paper) and $A_2 = 2$ (as either $b_1 = 1$ and $b_2 = 2$ or $b_1 = 2$ and $b_2 = 1$).

The simplest way to see this is that A_k are the Fibonacci numbers, and thus $A_{10} = 89$. But also, note that

$$A_{10} = A_9 + A_8 = 2A_8 + A_7 = 3A_7 + 2A_6 = \cdots = 34A_2 + 21A_1 = 89.$$

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