Georgia Tech HSMC 2006

Varsity Multiple Choice

Answers

1. Evaluate $\log_2\left(1+\frac{1}{2}\sum_{k=1}^{6}\binom{7}{k}\right)$.

   $\boxed{(C)\ 6}$

2. For any number x, we define x^+ and x^- such that $x^+ + x^- = |x|$ and $x^+ - x^- = x$. Find the sum $\sum_{j=1}^{100}[(2j)^+ - (2j+1)^-]$.

   $\boxed{(A)\ 10100}$

3. For all real numbers a, b and some fixed real number n, the operation $*$ is defined as $a * b = a^2 - b * n$. Given that $11 * 11 = 5$, what is the value of $20 * 3$?

   $\boxed{(B)\ 396}$

4. Suppose that x and y are real numbers that satisfy $\lfloor x \rfloor + \{y\} = 3.2$, and $\{x\} + \lfloor y \rfloor = 1.7$, where $\lfloor x \rfloor$ represents the greatest integer less than or equal to x, and $\{x\} = x - \lfloor x \rfloor$. Find the difference $x - y$.

   $\boxed{(D)\ 2.5}$

5. Isosceles $\triangle ABC$ is such that $|AB| = |AC| = \sqrt{\sin\theta}$, $|BC| = \sin\theta$, where θ is the measure of $\angle A$. Find the area of this triangle.

   $\boxed{(A)\ \frac{8}{25}}$

6. Define $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \dfrac{5}{1+31e^x}$. How many different integer values can $f(x)$ take?

   $\boxed{(C)\ 4}$

7. In Ms. Brakebill's calculus 2 class there are 20 students. Furthermore, we know there are 8 female students, 7 freshmen, and 8 math majors. We also know that 3 students are female and freshmen, 4 students are freshmen and math major, 3 students are female and math major, and 1 student who is a freshman, math-major female. How many students in Ms. Brakebill's class are neither female nor freshmen nor math major?

   $\boxed{(E)\ 6}$

8. Evaluate the sum

$$\sum_{n=1}^{\infty}[\arctan(n+1) - \arctan(n-1)].$$

   $\boxed{(D)\ \frac{3\pi}{4}}$

9. How many values $-\infty < x < +\infty$ satisfies $x\lfloor x \rfloor = 8$, where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x?

   $\boxed{(B)\ 1}$

10. How many functions $f : \mathbb{R} \to \mathbb{R}$ satisfy, for all real x, y, $f(x) + f(y) = xf(y)$

$\boxed{(B)\ 1}$

11. A friend has a stack of ten cards numbered 1 through 10. He shuffles the cards and you draw two cards. What is the probability that the cards are consecutive numbers?

$\boxed{(B)\ \frac{1}{5}}$

12. Points A, B, C, D are on the plane so that $\angle ABC + \angle ADC = 180^{\cdot}$. Perpendicular bisectors to AB and AD intersect at point P. If $|PC| = 17$, find $|PB|$.

$\boxed{(C)\ 17}$

13. If $x = \dfrac{1002}{1003}$, for what positive integer n is the following equation satisfied?

$$x + 2x + ... + nx = x^2 + 2x^2 + ... + (n+1)x^2.$$

$\boxed{(B)\ 2004}$

14. How many zeros are at the end of $\binom{100}{51}$?

$\boxed{(B)\ 2}$

15. How many points of positive integer coordinates are on the ellipse $2x^2 + 13y^2 = 213$?

$\boxed{(B)\ 1}$

16. At how many points do $y = \sin x$ and $y = \dfrac{1}{5\pi}x$ intersect?

$\boxed{(D)\ 11}$

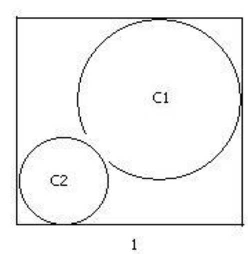
17. A 3-digit palindrome is a 3-digit number (not starting with 0) which reads the same backwards and forwards (for example, 171). Find the sum of all even 3-digit palindromes.

$\boxed{(B)\ 22000}$

18. How many pairs (a, b) of positive integer numbers are solution of the equation $a^2 - b^2 = 105$?

$\boxed{(E)\ 4}$

19. Given a square with sidelength 1, determine the radius of the smaller circle C_2 given C_1's radius is twice that of C_2's, and these circles are tangent to each other and the square.



$\boxed{(B)\ \dfrac{\sqrt{2}}{3(\sqrt{2}+1)}}$

20. A regular octahedron has volume $V \neq 1$. Its surface area can be written as $S = 2^a 3^b V^c$, where a, b, and c are rational numbers. Find $a + b + c$.

$\boxed{(A)\ \frac{5}{2}}$