Georgia Tech HSMC 2006

Varsity Proof

Solutions

1. The roots of $16(1-x) = \frac{1}{x}$ are x_1, x_2. The tenth digit after the decimal point of x_1 is 8; find the tenth such digit in x_2.

   Solution The equation $16(1-x) = \dfrac{1}{x}$ can be rewritten as

   $$16x^2 - 16x + 1 = 16(x - x_1)(x - x_2) = 0,$$

   therefore $x_1 + x_2 = 1 = .9999999\ldots$ and $x_1 x_2 = 1/16$. Then, both x_1 and x_2 are both positive and therefore the tenth decimal digit after the point of x_1 and the tenth decimal digit after the point of x_2 have add up to 9. Given this, the tength digit after the decimal point of x_2 is 1.

   $\square$

2. Among all pairs of $2k$ digit natural numbers a, b (in decimal representation) that have the same set of digits with all digits positive, what isthe largest possible value of $a - b$? Prove your results. For example, if $n = 2$ then the largest value is $91 - 19 = 72$.

   Solution: Note that if we give a set of digits $d_0,\ d_1, \ldots d_{2k-1}$, assuming $1 \le d_0 \le d_1 \le \cdots \le d_{2k-1} \le 9$, then, the greates value would be $a = d_{2k-1} d_{2k-2} \ldots d_1 d_0$ and the smallest would be $b = d_0 d_1 \ldots d_{2k-2} d_{2k-1}$. Thus

   $$\begin{aligned} a - b &= (10^{2k-1} d_{2k-1} + \cdots + 10 d_1 + d_0) \\ &\quad -(10^{2k-1} d_0 + \cdots + 10 d_{2k-2} + d_{2k-1}) \\ &= (10^{2k-1} - 1)(d_{2k-1} - d_0) + \\ &\quad (10^{2k-2} - 10)(d_{2k-2} - d_1) + \cdots + \\ &\quad (10^k - 10^{k-1})(d_k - d_{k-1}) \end{aligned}$$

   Therefore we are trying to maximize the numbers $d_{2k-1} - d_0,\ d_{2k-2} - d_1 \ldots d_{2k-1} - d_2$, what is obtained when $d_0 = d_1 = \cdots = d_{k-1} = 1$ and $d_k = \cdots = d_{2k-2} = d_{2k-1} = 9$, and therefore $a - b = \underbrace{8888\ldots8}_{k-1 \text{ times}} 7 \underbrace{1111\ldots1}_{k-1 \text{ times}} 2$.

   $\square$

3. Prove that, for integers $m \ge 1$, $(4m)!(m!)^4 > ((2m)!)^4$.

   Solution: Note that

   $$\begin{aligned} & (4m)!(m!)^4 > ((2m)!)^4 \\ \Leftrightarrow\ & \frac{(4m)!}{(2m)!(2m)!} > \frac{(2m)!}{m!m!} \cdot \frac{(2m)!}{m!m!} \\ \Leftrightarrow\ & \binom{4m}{2m} > \binom{2m}{m} \cdot \binom{2m}{m} \end{aligned}$$

   Now, suppose that we have a group of $4m$ people, $2m$ of which are males and the other $2m$ are females. There is exactly $\binom{4m}{2m}$ ways of choosing a subgroup of $2m$ people, and in some of those ways (not all) we will have exactly m males and m females in the subgroup of $2m$. Now, the number of ways of choosing exactly m males and m females would be $\binom{2m}{m}^2$, therefore

$$\binom{4m}{2m} > \binom{2m}{m}^2 \Rightarrow (4m)!(m!)^4 > ((2m)!)^4.$$

□

4. 2006 can be factored as $2 \times 17 \times 59$. Prove that for all $x \in \mathbb{R}$

$$\left(\frac{1003}{2}\right)^x + \left(\frac{118}{17}\right)^x + \left(\frac{34}{59}\right)^x \geq 2^x + 17^x + 59^x.$$

Remark: $2006 = 1003 \cdot 2 = 118 \cdot 17 = 34 \cdot 59$.

Solution: Note that by arithmetic-geometric mean we have that:

$$\frac{1}{2}\left[\left(\frac{1003}{2}\right)^x + \left(\frac{118}{17}\right)^x\right] \geq \left[\left(\frac{1003}{2}\right)^x \left(\frac{118}{17}\right)^x\right]^{\frac{1}{2}} = 59^x$$

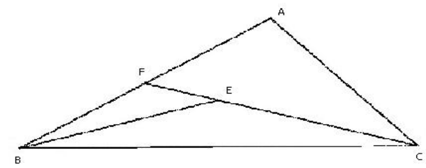
Similarly

$$\begin{aligned} \frac{1}{2}\left[\left(\frac{118}{17}\right)^x + \left(\frac{34}{59}\right)^x\right] &\geq 2^x \\ \frac{1}{2}\left[\left(\frac{1003}{2}\right)^x + \left(\frac{34}{59}\right)^x\right] &\geq 17^x \end{aligned}$$

Adding all of them together

$$\left(\frac{1003}{2}\right)^x + \left(\frac{118}{17}\right)^x + \left(\frac{34}{59}\right)^x \geq 2^x + 17^x + 59^x.$$

□



5. In $\triangle ABC$, $\angle ABC = 30°$, and $\angle ACB = 45°$. Let E be inside of this triangle in such a way that triangle EBC is isoceles (i.e. $|EB| = |EC|$), and $\angle EBC = 15°$. Denote by F the intersection of CE and AB. Show that $|AF| = |FB|$.

Solution Note that by Sine Law we have that

$$\begin{aligned} \frac{|AF| + |FB|}{\sin 45°} &= \frac{|AC|}{\sin 30°} \\ \frac{|AF|}{\sin 30°} &= \frac{|AC|}{\sin 45°} \end{aligned}$$

Then, solving the second equation for $|AC|$ and substituing it on the first we have that

$$|AF| + |FB| = \frac{\sin^2 45°}{\sin^2 30°}|AF| = 2|AF|$$

$$\Rightarrow |AF| = |FB|.$$

□