Ciphering Round Varsity League

High School Math Competition 2006

Georgia Institute of Technology

March 4^{th}, 2006

Problem #1

Problem

Find $a_1 + a_2 + a_3 + \cdots + a_9$ given that

$$(1 + x + x^2)^5 = a_0 + a_1 x + a_2 x^2 + \cdots + a_9 x^9 + a_{10} x^{10}.$$

Problem #1

Problem

Find $a_1 + a_2 + a_3 + \cdots + a_9$ given that

$$(1 + x + x^2)^5 = a_0 + a_1 x + a_2 x^2 + \cdots + a_9 x^9 + a_{10} x^{10}.$$

Answer

$$a_1 + a_2 + a_3 + \cdots + a_9 = 3^5 - 2 = 241$$

Problem #2

Problem

Find the prime factorization of $3^{10} + 3^{9} - 12$.

Problem #2

Problem

Find the prime factorization of $3^{10} + 3^9 - 12$.

Answer

$$3^{10} + 3^9 - 12 = 2^7 \cdot 3 \cdot 5 \cdot 41$$

Problem #3

Problem

Compute

$$\frac{1004}{2006} + \left(\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2005 \cdot 2006} \right).$$

Problem #3

Problem

Compute

$$\frac{1004}{2006} + \left(\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2005 \cdot 2006}\right).$$

Answer

$$\frac{1004}{2006} + \left(\frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \cdots + \frac{1}{2005 \cdot 2006}\right) = 1$$

Problem #4

Problem

Find the sum of the digits of $10^{2005} - 2006$.

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Problem

Find the sum of the digits of $10^{2005} - 2006$.

Answer

18038

Problem #5

Problem

If α and β are the two roots of the equation $2x^2 + 4x + 3 = 0$, find $\alpha^2 + \beta^2$.

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Answer

$$\alpha^2 + \beta^2 = 1$$

Problem #6

Problem

If $P \subseteq \{1, 2, 3, \ldots, 48, 49\}$ has the property that it does not have two distinct elements with sum divisible by 7, what is the maximal amount of elements P can have?

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Answer

22

Problem #7

Problem

If for all $x \neq 0$ one has that $x^{-1}f(-x) + f(x^{-1}) = x$, find $f(1)$.

Problem #7

Problem

If for all $x \neq 0$ one has that $x^{-1}f(-x) + f(x^{-1}) = x$, find $f(1)$.

Answer

$$f(1) = 1$$

Problem #8

Problem

If $a > 0$ and $a^2 + \frac{1}{a^2} = 7$, find $a^3 + \frac{1}{a^3}$.

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Problem

If $a > 0$ and $a^2 + \frac{1}{a^2} = 7$, find $a^3 + \frac{1}{a^3}$.

Answer

$$a^3 + \frac{1}{a^3} = 18$$

Problem #9

Problem

In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5,..., The ordered sequence with n terms equal to n, what is the 2006^{th} term in the sequence?

Problem #9

Problem

In the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5,..., The ordered sequence with n terms equal to n, what is the 2006^{th} term in the sequence?

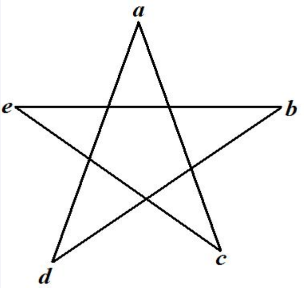
Answer

63

Problem #10

Problem

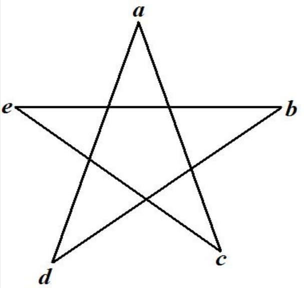
Find $a + b + c + d + e$ where each letter represents the measure of the angle indicated in the figure.



Problem #10

Problem

Find $a + b + c + d + e$ where each letter represents the measure of the angle indicated in the figure.



Answer

$$a + b + c + d + e = 180^\circ = \pi \text{ radians}$$

THE END