2005 Georgia Tech High School Mathematics Competition

Varsity Multiple-Choice Examination – Version B

SOLUTIONS

Problem 1: Calculate the expression: $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + ... + n \cdot n!$.

(A) $(n^2+n+1) \cdot n!$ (B) $(n+1)! - 1$ (C) $(n+2)! - n!$ (D) $(n!)^2 - 1$ (E) None of the above

Notice that $k \cdot k! = (k+1)! - k!$. Then,

$$\sum_{k=1}^{n} k \cdot k! = \sum_{k=1}^{n} ((k+1)! - k!) = (n+1)! - 1$$

Answer is (B).

Problem 2: What are the last two digits of 103^{2005}?

(A) 01 (B) 23 (C) 43 (D) 63 (E) None of the above

Notice that $103 \equiv 3 mod 100$, so the last two digits of 103^{2005} will be the same as the last two digits of 3^{2005}. Also, 3 and 100 are relatively prime (they have no common factors), so the order of 3 *mod* 100 will divide 100 (the order of 3 refers to the exponent of 3 that returns 1 modulo another integer). This method or a more trial and error method will show that $3^{20} \equiv 1 mod 100$, so $3^{2005} = (3^{20})^{100} \cdot 3^5 \equiv 3^5 = 243 \equiv 43 mod 100$. Answer is (C).

Problem 3: Solve the following linear system of equations for the value of x:

$$\begin{aligned} x + 2y + 3z &= 1 \\ 2x + 3y + 4z &= 2 \\ -x + y + 2z &= 1 \end{aligned}$$

(A) -2 (B) -1 (C) 0 (D) 1 (E) 2

Solution by any choice of methods gives $x = -1$, $y = 4$ and $z = -2$. Answer is (B).

Problem 4: How many positive integers $n < 1000$ satisfy $\lfloor \frac{n/3}{150} \rfloor = \frac{\lfloor n/3 \rfloor}{150}$, where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to x?

$(A)\ 4 \qquad (B)\ 6 \qquad (C)\ 8 \qquad (D)\ 9 \qquad (E)$ None of the above

Numbers that satisfy the relation are 1,2,450,451,452,900,901,902. Answer is (C).

Problem 5: Evaluate the term $\left(\frac{1+i}{2}\right)^{10}$, where $i = \sqrt{-1}$.

$(A)\ \dfrac{1-i}{4} \qquad (B)\ \dfrac{1}{8} \qquad (C)\ \dfrac{-1+i}{16} \qquad (D)\ \dfrac{i}{32} \qquad (E)$ None of the above

Use the polar form of $\frac{1+i}{2} = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}$ to perform the multiplication.

$$\left(\frac{1+i}{2}\right)^{10} = \left(\frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}\right)^{10} = \frac{1}{\sqrt{2}^{10}}e^{i\frac{10\pi}{4}} = \frac{i}{32}$$

Answer is (D).

Problem 6: We surveyed 83 freshman at Georgia Tech one semester and found that:

27 take English, 27 take American History, 55 take Calculus
12 take English and Calculus, 7 take English and American History
12 take Calculus and History, 2 take all three courses

How many students take NONE of these three courses?

$(A)\ 1 \qquad (B)\ 3 \qquad (C)\ 5 \qquad (D)\ 9 \qquad (E)$ None of the above

Use inclusion/exclusion principles (visually with a Venn diagram) to obtain the solution (let E = English, C = Calculus, H = American History)

$$\text{NONE} = \text{Total} - E - C - H + E\cap C + E\cap H + C\cap H - E\cap C\cap H = 83 - 27 - 55 - 27 + 12 + 7 + 12 - 2 = 3$$

Answer is (B).

Problem 7: Determine the value of x:

$$-7ln(\frac{9}{10}) + 2ln(\frac{24}{25}) + 3ln(\frac{81}{80}) = ln(x)$$

$(A)\ 8 \qquad (B)\ 6 \qquad (C)\ 4 \qquad (D)\ 2 \qquad (E)$ None of the above

Break all arguments into their prime factors after applying the exponent property of logs.

$$\ln(x) = \ln(\frac{3^{-14}}{2^{-7}5^{-7}} \cdot \frac{2^6 3^2}{5^4} \cdot \frac{3^{12}}{2^{12}5^3}) = \ln(2)$$

Answer is (D).

Problem 8: How many even numbers between 1000 and 9999 have all distinct digits?

(A) 2296 (B) 2298 (C) 2300 (D) 2302 (E) None of the above

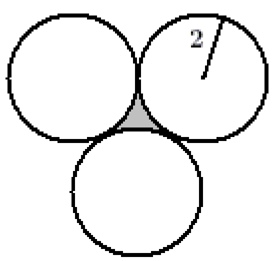
Even numbers end in 0,2,4,6, or 8. Numbers between 1000 and 9999 cannot have a 0 as the first digit. Therefore the number of even numbers between 1000 and 9999 with distinct digits is $9 \cdot 8 \cdot 7 \cdot 1 + 8 \cdot 8 \cdot 7 \cdot 4 = 2296$ to account for those numbers ending in zero as well as those ending with another even number. Answer is (A).

Problem 9: The equation $x^2 + 2x + y^2 + 6y + 6 = 0$ defines a circle. What is its radius?

(A) 2 (B) $\sqrt{6}$ (C) $\sqrt{14}$ (D) 6 (E) None of the above

Factoring the equation into $(x+1)^2 + (y+3)^2 = 4$ results in a radius of $\sqrt{4} = 2$. Answer is (A).

Problem 10: Three mutually tangent circles of equal radius two are shown in the figure below. What is the area of shaded portion between the three circles?



(A) $\sqrt{3} - \frac{\pi}{2}$ (B) $4\sqrt{3} - 2\pi$ (C) $\frac{4\sqrt{3} - \pi}{3}$
(D) $2\sqrt{6} - \pi$ (E) Not enough information given

Mutually tangent circles create an equillateral triangle with their centers. Therefore, the area of the shaded region is the area of that triangle, $\frac{1}{2} \cdot 4 \cdot (2\sqrt{3}) = 4\sqrt{3}$, minus the area of the circles inside the equillateral triangle, $3 \cdot \frac{1}{6} \cdot \pi \cdot 2^2 = 2\pi$.

Answer is (B).

Problem 11: The fundamental theorem of algebra states that a polynomial of order n, $p_n(x) = x^n + \alpha_{n-1}x^{n-1} + \alpha_{n-2}x^{n-2} + \ldots + \alpha_1 x + \alpha_0$, has exactly n roots. If all of the coefficients, $(\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_1, \alpha_0)$, are real numbers, which of the following statements are true (mark all that apply)?

(A) If n is even, there must exist at least one real root.
(B) If n is odd, there must exist at least one real root.
(C) If n is even, there must be at least one complex root.
(D) The sum of the coefficients, $\sum_{k=1}^{n-1} \alpha_k$, must be positive.
(E) All complex roots will occur in conjugate pairs.

Both (B) and (E) are true. A counterexample to (A) is $p_2(x) = x^2 + 1$, to (C) is $p_2(x) = x^2 - 1$, and to (D) is $p_1(x) = x - 1$.

Problem 12: What is $\sum_{n=0}^{360} \sin(\frac{n2\pi}{360})$?

(A) 0 $\quad (B)\ \frac{1}{2} \quad (C)$ 1 $\quad (D)\ \frac{\pi}{2} \quad (E)$ None of the above

For every index $n \neq 180$, consider the index $360 - n$. Then $\sin(\frac{n2\pi}{360}) + \sin(\frac{(360-n)2\pi}{360}) = 0$ shows that the sum can be reduced to $\sin(0) + \sin(\pi)$, which is zero. Answer is (A).

Problem 13: What is the minimum value of $f(n) = 2n^2 + 3n + 1$, where n can be any real number?

(A) 1 $\quad (B)\ \frac{-3}{4} \quad (C)\ \frac{-1}{8} \quad (D)\ \frac{-1}{16} \quad (E)$ None of the above

The minimum value occurs at $n^* = -\frac{3}{4}$ (which can be obtained via calculus or geometric properties of parabolas), giving $f(n^*) = -\frac{1}{8}$. Answer is (C).

Problem 14: What is the sum of all the digits in the sequence $1, 2, 3, 4, \ldots, 2003, 2004, 2005$?

(A) 28,027 $\quad (B)$ 29,327 $\quad (C)$ 35,615 $\quad (D)$ 37,137 $\quad (E)$ 39,213

Consider the first 1998 numbers, taking the first and last as a pair to create a digit sum equal to that of 1999. Continue this pattern [(1,1998),(2,1997),...,(999,1000)] until you have $\frac{1998}{2} = 999$ pairs that each have a digit sum equal to that of 1999. The sum of all digits is then $(999+1)\cdot(1+3\cdot 9)+(6\cdot 2)+(1+2+3+4+5)=28027$. Answer is (A).

Problem 15: What is the value of $\cos(\frac{5\pi}{12})$?

$(A)\ \dfrac{1+\sqrt{3}}{2\sqrt{2}}$ $(B)\ \dfrac{-1+\sqrt{3}}{2\sqrt{2}}$ $(C)\ \dfrac{1-\sqrt{3}}{2\sqrt{2}}$ $(D)\ \dfrac{-1-\sqrt{3}}{2\sqrt{2}}$ (E) None of the above

$$\cos(\frac{5\pi}{12})=\cos(\frac{\pi}{4}+\frac{\pi}{6})=\cos(\frac{\pi}{4})\cdot\cos(\frac{\pi}{6})-\sin(\frac{\pi}{4})\cdot\sin(\frac{\pi}{6})=\frac{\sqrt{2}}{2}\cdot\frac{1}{2}-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}=\frac{1-\sqrt{3}}{2\sqrt{2}}$$

Answer is (C).

Problem 16: Convergence of the sequence $X_n=(x_0,x_1,x_2,\ldots)$ implies that the quantity $|x_{n+1}-x_n|$ tend to zero in the limit as $n\to\infty$.

The recurrance relation, $z_n=\frac{3}{4}\cdot z_{n-1}+3$, is a convergent sequence. Determine the value that it converges to.

(A) 3 $(B)\ \dfrac{9}{4}$ (C) 12 (D) 15 (E) None of the above

If Z_n is convergent, then $|z_{n+1}-z_n|$ tends to zero for sufficiently large n. Therefore, we rewrite the recurrence relation as $z_n=\frac{1}{4}z_n+\frac{3}{4}z_n=\frac{3}{4}z_{n-1}+3$. Subtracting $\frac{3}{4}z_{n-1}$ from both sides, we obtain

$$\frac{1}{4}z_n+\frac{3}{4}(z_n-z_{n-1}) \quad\rightarrow\quad \frac{1}{4}z_n=3$$

Therefore, the sequence Z_n converges to $\frac{3}{1/4}=12$. Answer is (C).

Problem 17: Given the standard coins issued by the U.S. treasury (1-cent, 5-cent, 10-cent, 25-cent, and 50-cent), what is the largest amount of money a person can have in coins and be unable to make change for a dollar?

(A) 99 cents (B) 1.09 dollars (C) 1.14 dollars (D) 1.19 dollars (E) 1.24 dollars

Largest coin collection without being able to make change for a dollar is 3 quarters (or one half-dollar and one quarter), 4 dimes, and 4 pennies for a total of 1.19 dollars. Answer is (D).

Problem 18: An element x is said to be in the **null space** of linear operator, A, if $A\cdot x=0$. If A and B are taken to be two nxn matrices (linear operators) and x is a length-n vector belonging to the nullspace of A, which of the following are true? Mark all that apply.

(A) $(A \circ B)x = 0$ (B) $(B \circ A)x = 0$
(C) The null space of $(A \circ B)$ is a subset of the null space of A
(D) The null space of $(B \circ A)$ is a subset of the null space of A
(E) The null space of $(A \circ B)$ is the same as the null space of $(B \circ A)$

(B) and (D) are correct. Note that x in the null space of A implies $Ax = 0$, but it does not tell us anything about Bx. (A) could be true, but is not necessary, while (B) is proven by $(B \circ A)x = B(Ax) = B \circ 0 = 0$. (C) could be true, but is also not necessary, whereas (D) recognizes that anything in the null space of A will also be in the null space of $(B \circ A)$, independent of B (B may have additional elements in its null space, so we can only speak of a subset, not an identical set). (E) should follow from the explanations of (C) and (D).

Problem 19: Suppose you are presented with a 9x9 grid, where each of the sections contains exactly one yellow jacket. After one time unit, every yellow jacket is supposed to move to an adjacent space (they may pass each other). How many different ways can this be achieved?

(A) 1 (B) 9! (C) $9 \cdot 8$ (D) $\dfrac{9!}{2!}$ (E) None of the above

On any square grid having an odd number of sides, moving every piece to an adjacent square will be impossible. Consider a checker-board pattern where every other space is colored, alternating between white and gold. If the first space is white, then for an n-sided board (n is odd) there will be $\frac{n^2+1}{2}$ white squares and $\frac{n^2-1}{2}$ gold squares. Since every yellow jacket must move from a white to gold or gold to white square, we must have an equal number of squares to move all the yellow jackets, it is impossible to move them all. Answer is (E).

Problem 20: Find the number of distinguishable permutations of the letters in the word:

HELLUVANENGINEER

(A) 10^{16} (B) $\dbinom{16}{10}$ (C) $\dfrac{16!}{2!3!4!}$ (D) 16! (E) None of the above

"I'm a ramblin' wreck from Georgia Tech and a helluvan engineer!" Total number of permutations is 16!, but this overcounts the possible number of distinct permutations since the 2 L's, 3 N's, and 4 E's are in themselves indistinguishable. Answer is (C).